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XLI. *A new form of Amperemeter and Voltmeter with a Long Scale.* *By* Benjamin Davies *. (Communicated by Prof. Lodge.)

These measuring-instruments, which are intended for continuous currents, are of the so-called d'Arsonval type, and their chief feature is their long and uniformly divided scale. Mainly on account of these advantages, I am of opinion that the instruments may become generally useful for electrical measurement on switch-board or on the bench, and particularly in the laboratory for purposes of experiment and research.

With the exception of instruments depending on magnification, these (so far as I am aware) are the only long-range ones yet designed ; a brief description of them may, therefore, not be unwelcome †.

It is evident that a long scale is a desirable addition to the best of instruments, provided the extra length can be produced without increasing the waste-power of the meter, or without reducing the deflecting moment. It is evident that this is a question of design only, and the instruments described below have a satisfactorily large deflecting moment, although the scale covers an arc varying from 210° to 230°. By sacrificing a little of the deflecting force, or by increasing the voltage drop at the terminals of the moving coil, an instrument may be constructed with a scale-length equal to an arc of 270°.

The magnetic circuit also has one distinct point in its favour, viz., its single air-gap. This air-gap may be as narrow as $\frac{1}{16}$ of an inch, reducing the demagnetizing force to a small quantity.

The permanent magnet and the magnetic circuit generally may have many forms, each having its own advantages and

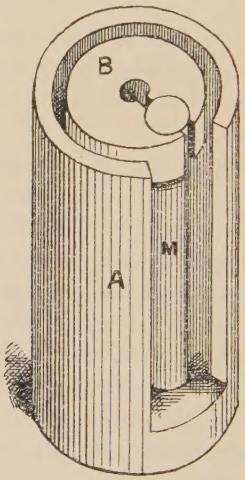
* Read February 10, 1899.

† After this paper was read I found I was by no means the first in the field as regards "long range." Professor Ayrton kindly sent me a sketch of an amperemeter by M. Carpentier, built on precisely the same principle as those mentioned in this paper. The Carpentier instrument, however, is not of good design, resulting, I should imagine, in an excessive drop of voltage at the terminals of the moving coil. The Carpentier meter, I am told, was brought out in 1889. The first form of mine was made in 1895.

disadvantages. The construction of the earliest form is shown in fig. 1. A is an iron case, either wrought or cast. B is the soft iron polepiece, which is carried and maintained in position by the permanent steel magnet M.

From a physical point of view, this combination forms a very good magnetic circuit, for the reluctance of the air-gap between polepiece and box is very small, owing to its extensive area. The relative thickness of the gap shown in the figure is much greater than it is in the actual instrument.

Fig. 1.



The form of Magnetic Circuit first employed.

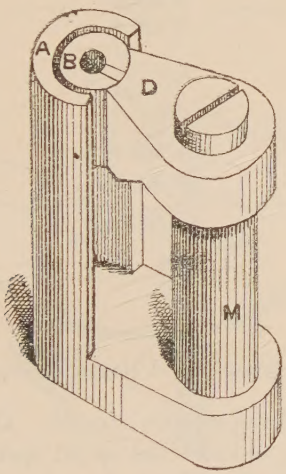
When this form was tried it was found to have some defects on practical grounds ; the attachment and adjustment of the moving system became a difficult matter. Once made, however, an instrument having this form of magnetic circuit is not at all a bad one, and of all forms it offers the longest range.

The moving coil was rectangular in shape, and one side of it—forming the axis around which it rotated—passed through the central hole in B, while the opposite side moved in the narrow air-gap. This form of magnetic circuit (fig. 1) is not unlike that adopted by Mr. W. Hibbert in his apparatus for

a standard magnetic field, except that in mine the steel magnet is eccentric *.

The next form is shown in fig. 2. This is much more easily manipulated, and the difficulty of adjusting the movable system is considerably reduced. M is the steel magnet, A and B D are the two wrought-iron poles. The frame supporting the moving system is in this case mounted on the polepiece A.

Fig. 2.



Another form of Magnetic Circuit.

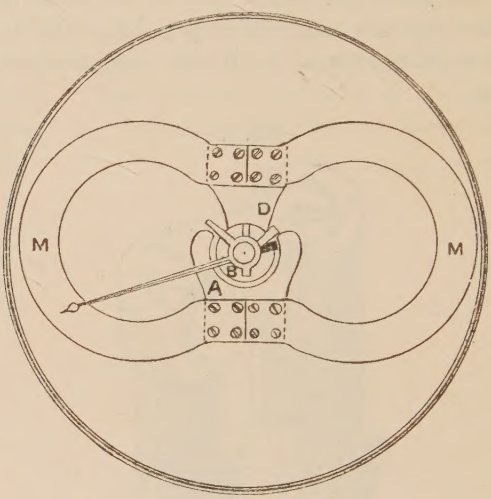
In fig. 3 we have a plan of a still better form considered from a practical point of view, though less elegant and compact than its predecessors. The construction is evident from the figure. M M are the magnets, now of the horse-shoe shape, A and B D the polepieces. This form of circuit has been adopted mainly in order to increase the induction. The previous forms do not lend themselves to the use of large cross-sections of steel without becoming inconvenient in shape. In this pattern it is evident that any desired quantity of steel may be used.

On the soft-iron cylinder B is mounted the brass frame carrying the entire moving system. The air-gap is adjusted

* See Proc. Phys. Soc. vol. xi. p. 303.

for uniformity of induction by means of the D portion of the polepiece, the details of which for the sake of simplicity are not given in the figure.

Fig. 3.



The latest form of Magnetic Circuit.

Considering next the electric circuit. The main parts of this are the moving coil and the spiral springs. The coil is wound on an aluminium rectangle, which serves the double purpose of a frame to stiffen the coil and a circuit of extremely low resistance for damping.

In the voltmeter the moving coil contains some 100 or 200 turns of the finest wire, while the amperemeter coil has some 20 turns of a moderately thick wire.

In series with the voltmeter moving coil there is placed in the base of the instrument a resistance-coil, on the value of which depend the readings of the meter.

In parallel with the amperemeter-coil is placed a short length of conductor of a similar material to that of the moving coil. Through this shunt flows the main current in all meters except the centiampere and milliampere ones.

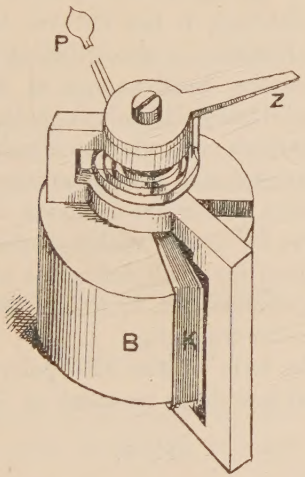
The current is led into and out of the moving coil by the spiral springs, which also provide the opposing or controlling force. To partially counterpoise the coil it is placed—with

respect to the axis—opposite the pointer. The coil and pointer are shown in position in the plan given in fig. 3.

A coil of this kind, swung on one of its sides, was used long ago by Lord Kelvin in his Siphon Recorder. It is also used at the present time by Professor Ayrton in the now well-known Astatic Station Voltmeter.

Fig. 4 gives a better view of coil, springs, and pointer, and the general method of mounting the entire moving system on the iron cylinder. K is the coil, P the pointer, and Z the

Fig. 4.



The central polepiece to which is attached the brass frame supporting the coil, pointer, and spiral springs.

arm for zero adjustment. There is here a compactness that becomes important in practice, for by loosening or tightening one screw the moving system with its frame can be removed bodily and placed in position again with ease, and without interfering with the magnetic circuit. The method also renders the balancing of the coil an easy matter.

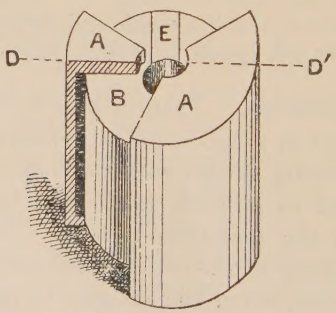
In all the meters, when the air-gap is made uniform in thickness, the induction density in the gap has practically a constant value throughout its entire length (except in the immediate neighbourhood of the two ends owing to the effect of prominences). This is the case even for the magnetic

circuits of figs. 1 and 2, where the induction is rather low, viz. (about) 400 per sq. cm. in the gap. With a magnetic flux of 400 the reluctance of the air is about 300 times that of the iron; so that the lines distribute themselves pretty uniformly on the polar surfaces if the intervening gap is made uniform in thickness. For the higher inductions of 800 or 1000 (for which the magnetic circuit of fig. 3 is designed) the magnetic force in the gap becomes still more uniform, for now we have the reluctance of the air 500 times that of iron.

There is another arrangement of polepieces which I would like to describe, although it has not yet been made into an instrument. It furnishes a great deflecting moment, but it has the rather serious disadvantage of being complex in design and—I am afraid—difficult to build. This arrangement, like the previous ones, may be attached either to a bar or a horse-shoe magnet, and may be arranged to give with a single moving coil a maximum deflexion of 220°, or it may be designed to give 80° or 90° with a double or twin coil. It may also be designed for small deflexions with a spot of light to be used as a galvanometer, in which case there is no difficulty at all in the construction.

Fig. 5 shows the form of the 220° polepieces without the magnet. The coil is entirely enclosed in iron, and its three

Fig. 5.



sides are active. In all other respects the arrangement is precisely the same as the previous ones.

Let fig. 6 represent a vertical section through D D′ in fig. 5, and let fig. 7 be the movable coil.

Fig. 6.

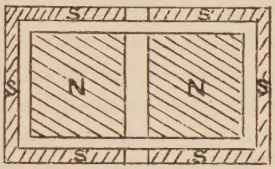
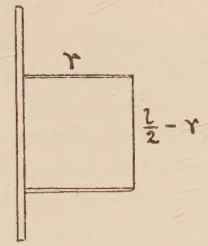


Fig. 7.



The lines of force cut the coil perpendicularly at all points. Let l be the length of the periphery of the coil, r the radial side, and n the number of turns in it. Considering one of the horizontal sides of the coil with a current c flowing in it, the moment of force acting at a distance x on an element dx is

$$= \mathrm{H}n\,c\,x\,dx.$$

The moment acting on the vertical side is

$$= \mathrm{H}nc\left(\frac{l}{2} - r\right)r,$$

H being the intensity of the magnetic field.

Therefore, the total deflecting moment M acting on the coil is

$$= \mathrm{H}\,nc\left[\left(\frac{l}{2} - r\right)r + 2\int_0^r x\,dx\right].$$

So $\mathrm{M} = \frac{1}{2}nc\mathrm{H}lr$.

Unlike the old coil (that considered in the first part of this paper) this new coil has a deflecting moment increasing always in proportion to its radius if the magnetic field is kept constant. The moment is therefore a maximum when r is as nearly $\frac{1}{2}l$ as the constructional details of the instrument will permit. For the old coil the maximum moment is got when $r = \frac{1}{4}l$, *i. e.*, when the coil is a square. I offer the description of this instrument partly on account of its leading to (what is to me) a novel form of a ballistic galvanometer.

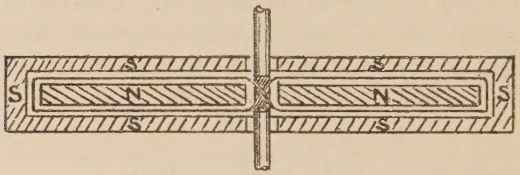
There may be more serious difficulties in the way of constructing a galvanometer on this plan than I at present anticipate.

Ballistic Galvanometer.

We evidently have in the arrangement shown in figs. 5, 6, and 7 favourable conditions for a ballistic galvanometer, since the deflecting moment and the moment of inertia of the coil attain their maximum values for the same value of the radius. The period of swing and the deflecting moment may therefore be increased simultaneously without making any addition to the weight of the moving system.

Let fig. 8 be a section through the polepieces of a ballistic

Fig. 8.



arrangement on the above plan, the coil completely embracing the central polepiece NN, and the outer polepiece S completely surrounding both. Here, of course, there is to be no aluminium frame for damping.

To compare the sensibility of this coil with another coil of the same mass, *effective area*, and number of turns, wound and suspended in the usual mode, let fig. 9 represent one, and

Fig. 9.

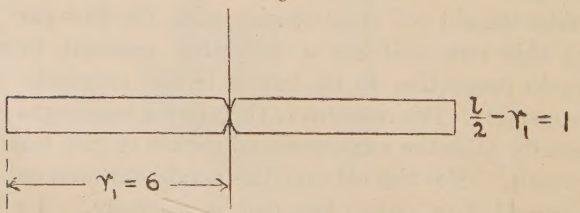
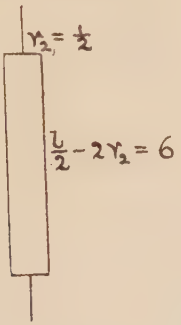


fig. 10 the other coil. Let $r_1 = 6$ in the first case, and let the depth of the coil be 1. In the second, let $r_2 = \frac{1}{2}$ and the

depth of the coil $=6$. Considered alone, the two coils are therefore similar in all respects. They are moreover supposed to be suspended in two fields of equal intensity, and to convey the same current.

Fig. 10.



For the sake of symmetry the wire in figs. 8 and 9 has to be divided between two rectangles, which, however, does not affect the argument.

The suspension in both cases having to bear the same load, the relative sensibilities of the two arrangements for continuous currents will therefore be as their deflecting-moments directly.

For fig. 9 the moment

$$M_1 = \tfrac{1}{2} ncH l r_1,$$

and for fig. 10

$$M_2 = 2ncH \left(\frac{l}{2} - 2r_2\right) r_2,$$

so that

$$\frac{M_1}{M_2} = \frac{lr_1}{4\left(\frac{l}{2} - 2r_2\right) r_2} = 7 ;$$

therefore $M_1 = 7M_2$ for steady currents.

Time of swing of the two coils.—The time of the swing varies with the $\sqrt{\text{moment of inertia.}}$

In fig. 9 the moment of inertia is

$$I_1 = m r_1^2 \left(\frac{l}{2} - \frac{r_1}{3}\right),$$

and for fig. 10

$$I_2 = 2mr_2^2\left(\frac{l}{2} - \frac{4r_2}{3}\right)$$

where m stands for the mass of unit length of the periphery.

Therefore

$$\frac{I_1}{I_2} = \frac{r_1^2(3l - 2r_1)}{2r_2^2(3l - 8r_2)}.$$

Hence for the particular size of coils under consideration,

$$\frac{I_1}{I_2} = \frac{1080}{19}.$$

So that $$I_1 = 57\, I_2 \text{ nearly.}$$

Therefore, the time of swing of the coil in fig. 9 will be 7·5 times that for the coil in fig. 10, and the deflecting-moment or the sensitiveness of the coil in fig. 9 for continuous currents will be 7 times that of the coil in fig. 10.

Now as regards the "throw." The relative throws of the two coils for the same quantity of electricity will be

$$= \frac{M_1}{M_2} \cdot \sqrt{\frac{I_2}{I_1}} = \frac{7}{7{\cdot}5} \text{ about.}$$

So that we have for the two coils throws of nearly equal magnitudes, although one coil moves 7·5 times as slowly as the other.

It is almost needless for me to mention that I have been greatly assisted during the experimental period of the measuring instruments described above, and I wish to express my gratitude to Professor Oliver Lodge not only for his suggestions and criticisms but also for his kind encouragement at all times. I am also greatly indebted to Dr. Alexander Muirhead for much help in practical details, and for the considerable experimental facilities he has always been ready to present.

Discussion.

Prof. Ayrton said the instruments appeared to be very successful; he could bear witness of their value particularly as regards the length of range. The general principle by which long range was to be obtained on moving-coil, portable instruments was developed some ten years ago by

M. Carpentier, of Paris, who used a central magnet surrounded by a concentric hollow cylinder, with only one side of the coil in the magnetic gap between them; but it was not then a portable form of instrument, for the coil was suspended. Prof. Ayrton and Mr. Mather had worked in this direction in their "astatic station voltmeter." In that instrument there were three magnetic circuits arranged to give astaticism; this was described in 1892 or 1893.

XLII. *Note on the Source of Energy in Diffusive Convection.*

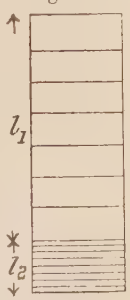
By Albert Griffiths, *M.Sc.* (*Vic.*), *A.R.C.S.* (*Lond.*).*

At the conclusion of a paper on "Diffusive Convection"† the author, partly in the hope of producing a discussion, asked certain questions relating indirectly to the source of energy in the apparatus under consideration.

After the publication of the paper in the Philosophical Magazine, Prof. FitzGerald made some remarks on it in 'Nature,' and gave a concise account of the actions at work. He pointed out, what was already known to the author, that there is a tendency towards cooling when diffusion causes the rise of the centre of gravity. Stimulated by Prof. FitzGerald's interest in the work, the author has taken advantage of the Christmas vacation to study the question in some detail.

On the Fall of Temperature when Diffusion occurs upwards if the Solution has a Density greater than that of Water.

Fig. 1.



Consider a vertical cylinder of length (l_1+l_2) and sectional area A. Let d_2 equal the density of length l_2, and d_1 the density of length l_1. Let it be assumed that there is no change of volume when the two liquids mix.

Taking a plane through the bottom of the cylinder as one where bodies at rest possess zero potential energy, it can readily be seen that before mixing, through diffusion or otherwise,

* Read March 10, 1899.

† Phil. Mag. ser. 5. vol. xlvi. p. 453 (1898); Proc. Phys. Soc. *ante*, p. 230.

the potential energy

$$= d_2 \mathrm{A} g \frac{l_2^2}{2} + d_1 \mathrm{A} g l_1 \left(l_2 + \frac{l_1}{2} \right).$$

The liquids also possess energy due to the heat they contain.

Let S_1 = thermal capacity of unit mass of liquid of density d_1.
S_2 = „ „ „ solution „ d_2.
S = „ „ „ mixture.
t_0 = temperature (absolute scale) before mixing.
t_1 = „ „ after „

The *energy*, before mixing, *in the form of heat* equals

$$\int_0^{t_0} d_1 \mathrm{A} l_1 \mathrm{S}_1 dt + \int_0^{t_0} d_2 \mathrm{A} l_2 \mathrm{S}_2 dt.$$

The combined potential and thermal energy equals

$$d_2 \mathrm{A} g \frac{l_2^2}{2} + d_1 \mathrm{A} g l_1 \left(l_2 + \frac{l_1}{2} \right) + \int_0^{t_0} d_1 \mathrm{A} l_1 \mathrm{S}_1 dt + \int_0^{t_0} d_2 \mathrm{A} l_2 \mathrm{S}_2 dt.$$

Let mixing take place without the addition or withdrawal of heat. Neglecting the heat of combination, the expression for the combined energy now equals

$$\frac{g(d_1 \mathrm{A} l_1 + d_2 \mathrm{A} l_2)(l_1 + l_2)}{2} + \int_0^{t_1} (d_1 \mathrm{A} l_1 + d_2 \mathrm{A} l_2) \mathrm{S} dt.$$

Equating the two expressions, dividing throughout by A, and rearranging,

$$\frac{g l_1 l_2 (d_1 - d_2)}{2} + \int_0^{t_0} d_1 l_1 \mathrm{S}_1 dt + \int_0^{t_0} d_2 l_2 \mathrm{S}_2 dt = \int_0^{t_1} (d_1 l_1 + d_2 l_2) \mathrm{S} dt.$$

If we make the hypothesis that S_1, S_2, and S are constants, then

$$\frac{g l_1 l_2 (d_1 - d_2)}{2} = \mathrm{S}(d_1 l_1 + d_2 l_2) t_1 - d_1 l_1 \mathrm{S}_1 t_0 - d_2 l_2 \mathrm{S}_2 t_0.$$

This equation indicates that $(t_0 - t_1)$, the fall in temperature, depends in some way on the height through which the

dissolved substance diffuses. The preceding was worked out long before I published my paper. In what follows, the influence of Prof. FitzGerald will perhaps be shown.

If we wish to study the fall of temperature due to the rise of matter by diffusion in a theoretical manner, we may assume that the capacity for heat of the mixture is the sum of the capacities for heat of the components, and that there is no heat of combination.

Let m = total mass of the two liquids.
s = thermal capacity of unit mass of mixture.
h = rise of C.G.
g = acceleration due to gravity.
f = the fall of temperature.

Then, since the potential energy gained equals the capacity of the liquids multiplied by the fall of temperature,

$$mgh = msf.$$

Hence

$$f = \frac{gh}{s}.$$

It may be pointed out that this equation indicates a limit to the height to which diffusion (with ultimate uniform distribution) can spread. Thus if s be a constant, the above equation holds whatever the fall in temperature; and if the temperature be initially 0° C., f cannot be greater than 273° C., and h cannot be greater than $273\,s/g$.

Let $s = 42{,}350 \times g$ (the capacity of a gram of water), then h cannot be greater than $273 \times 42{,}350$ cms.; that is 115 kilometres approximately. It may be noted that h is not the height at which diffusion occurs, but the rise in the centre of gravity. In a given case the height at which diffusion occurs can readily be expressed in terms of h.

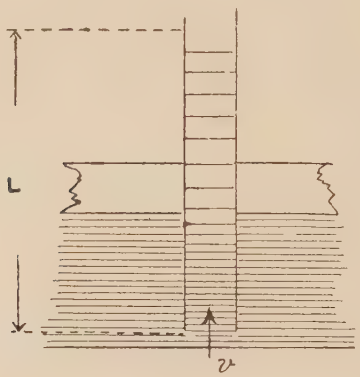
The Absorption of Heat in a simple case when there is a combination of Diffusion and Motion.

In the figure there is a representation of a vertical tube of length L.

Let us suppose that the lower end is continuously in contact

with an aqueous solution of concentration T, and that its upper extremity is in contact with pure water.

Fig. 2.



Let v = velocity of liquid up the tube.
A = sectional area of the tube.

With certain assumptions, it is shown in the paper on "Diffusive Convection" that the difference of pressure between the bottom and the top of the tube equals

$$g\mathrm{L} + \frac{g\mathrm{TL}}{1-e^{-\frac{v\mathrm{L}}{k}}} - \frac{kg\mathrm{T}}{v}.$$

The external work done on the cylinder per second equals the product of the volume which crosses any section and the difference of pressure; *i. e.*, it equals

$$v\mathrm{A}\left(g\mathrm{L} + \frac{g\mathrm{TL}}{1-e^{-\frac{v\mathrm{L}}{k}}} + \frac{kg\mathrm{T}}{v}\right)$$

$$= v\mathrm{A}g\mathrm{L} + \frac{v\mathrm{A}g\mathrm{TL}}{1-e^{-\frac{v\mathrm{L}}{k}}} + kg\mathrm{TA}.$$

The work performed per second consists of two parts—

(1) A quantity of water of volume vA is raised a height L.

(2) A quantity of dissolved substance (see "Diffusive Convection," Section II.) of amount $\frac{vAT}{1-e^{-\frac{vL}{k}}}$ is raised a height L.

Let H = the heat absorbed in unit time; we have:—
External work done − H = work performed: hence

$$vAgL + \frac{vAgTL}{1-e^{-\frac{vL}{k}}} + kgTA - H = vAgL + \frac{vATgL}{1-e^{-\frac{vL}{k}}},$$

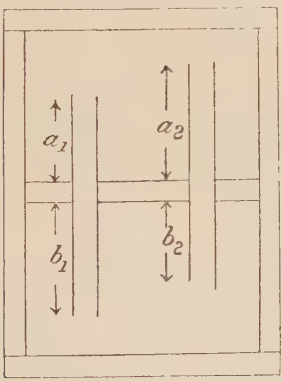
and

$$H = kgTA.$$

Thus the heat absorbed through the agency of the tube is independent of the height of the tube and of the velocity of flow.

Calculation of the Rate at which Work is done by the Apparatus described in "Diffusive Convection."

Fig. 3.



The apparatus to be studied is illustrated in fig. 3.

Let L_1 = length of left-hand tube.
L_2 = „ right „
A = area of each tube.
c = thickness of the diaphragm.
a_1 = difference of level between the top of L_1 and the top of the diaphragm.
a_2 = difference of level between the top of L_2 and the top of the diaphragm.

b_1 = distance between the bottom of L_1 and the bottom of the diaphragm.

b_2 = distance between the bottom of L_2 and the bottom of the diaphragm.

The work done by the apparatus is equal to the volume which crosses a section in unit time multiplied by what in Section II. of "Diffusive Convection" is called "the pressures tending to produce circulation."

The work done per second equals

$$vA\left[\left\{gL_2+\frac{gTL_2}{1-e^{\frac{vL_2}{k}}}+\frac{kgT}{v}+(b_1-b_2)(1+T)g\right\}-\left\{(a_2-a_1)g+gL_1+\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}-\frac{kgT}{v}\right\}\right];$$

or noting that $L_1=a_1+c+b_1$, $L_2=a_2+c+b_2$, it equals

$$vA\left\{\frac{2kgT}{v}+\frac{gTL_2}{1-e^{\frac{vL_2}{k}}}-\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}+g(b_1-b_2)T\right\}.$$

It may be mentioned that the work done equals zero when $v=0$ (in this case there is an impervious obstruction to the flow); and when

$$\frac{2kgT}{v}+\frac{gTL_2}{1-e^{\frac{vL_2}{k}}}-\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}+g(b_1-b_2)T=0$$

(if $\frac{vL}{k}$ is small), *i. e.* when

$$v=\frac{6k(\overline{a_2-a_1}+\overline{b_1-b_2})}{L_1^2+L_2^2}.$$

(In this case there is no obstruction whatever to the flow.)

The work done by the apparatus is a maximum when v is approximately half the latter value. Work is done on the apparatus when v is negative, or v is greater than

$$6k(\overline{a_2-a_1}+\overline{b_1-b_2})/(L_1^2+L_2^2).$$

Calculation, by means of Energy Equations, of the Rate at which Work is done.

Before tackling the problem, it will be well perhaps to make some preliminary remarks with regard to what occurs in the upper and lower compartments. The dissolved substance is being continually carried to the top of L_1, for example, and falling from there to the bottom of the upper compartment. The heat generated by the fall equals the product of the weight transmitted into the fall.

On the other hand, when the dissolved substance is abstracted from the bottom, the weakened solution rises to the top of the lower compartment. Diffusion into the weakened solution, whilst it is rising, will doubtless occur, *but the action can be made negligible by taking* g *very large*, in which case the weakened solution rises very rapidly.

If $d=1+t$ (where d=density and t=concentration), it can readily be shown that the heat generated equals the *weight transmitted* into the bottom of the tube *by diffusion* multiplied by the rise of the weakened solution.

The apparatus sketched in the preceding section will now be studied.

The quantity transmitted through L_1 into the upper compartment equals

$$\frac{v\mathrm{AT}}{1-e^{-\frac{vL_1}{k}}}.$$

The heat produced by the fall of the substance transmitted to the bottom of the upper compartment equals

$$\frac{v\mathrm{AT}ga_1}{1-e^{-\frac{vL_1}{k}}}.$$

The quantity transmitted into the bottom of L_1 *by diffusion* equals

$$\frac{v\mathrm{AT}}{1-e^{-\frac{vL_1}{k}}}-v\mathrm{AT}=\frac{v\mathrm{AT}e^{-\frac{vL_1}{k}}}{1-e^{-\frac{vL_1}{k}}}.$$

The energy produced by the rise of the diluted solution equals

$$\left(\frac{vATge^{-\frac{vL_1}{k}}}{1-e^{-\frac{vL_1}{k}}}\right)b_1.$$

Proceeding in this way, and making some algebraic transformations, it can readily be shown that the heat produced in this way when we consider the two tubes equals

$$vAT\left\{\frac{a_1+b_1e^{-\frac{vL_1}{k}}}{1-e^{-\frac{vL_1}{k}}}-\frac{a_2+b_2e^{\frac{vL_2}{k}}}{1-e^{\frac{vL_2}{k}}}\right\}.$$

One effect of the operations just considered is to reduce the quantity of the dissolved substance at the top of the lower compartment by a certain amount, and to place the same amount at the bottom of the upper compartment.

The potential energy gained per second is the product of the weight of the total amount transmitted and the thickness of the diaphragm. Thus the potential energy gained per second equals

$$g\left(\frac{vAT}{1-e^{-\frac{vL_1}{k}}}-\frac{vAT}{1-e^{\frac{vL_2}{k}}}\right)c.$$

Let W = rate at which work is done *by* the apparatus.

D = rate at which heat is absorbed due to diffusion along the tubes.

F = rate at which heat is produced by the fall of the dissolved substance in the upper compartment, and the rise of weakened solution in the lower.

V = rate at which potential energy is gained by the carriage of dissolved substance from the bottom to the top of the diaphragm.

Then

$$W = D - F - V$$

$$= 2kATg - vAT\left\{\frac{a_1+b_1e^{-\frac{vL_1}{k}}}{1-e^{-\frac{vL_1}{k}}}-\frac{a_2+b_2e^{\frac{vL_2}{k}}}{1-e^{\frac{vL_2}{k}}}\right\}$$

$$-\left(\frac{vAT}{1-e^{-\frac{vL_1}{k}}}-\frac{vAT}{1-e^{\frac{vL_2}{k}}}\right)c.$$

Noting that $a_1+c=L_1-b_1$, $a_2+c=L_2-b_2$, it can be shown algebraically that

$$W=vA\left\{\frac{2kgT}{v}+\frac{gTL_2}{1-e^{-\frac{vL_2}{k}}}-\frac{gTL_1}{1-e^{-\frac{vL_1}{k}}}+g(b_1-b_2)T\right\}.$$

This is exactly the same result as that obtained in an earlier section.

General Explanation of the Loss of Heat when Work is done by the Apparatus.

For simplicity let the thickness of the diaphragm be zero.

Transmission of dissolved substance along the tubes is accompanied by a fall of substance in the upper compartment, and a rise of weakened solution in the lower compartment, both of which actions produce heat. The rate of production of heat depends on the rate of flow of the liquid.

The *greater* the rate at which work is done by the apparatus, the less the rate of production of heat.

The rate of absorption of heat due to transmission of dissolved substance through the tubes is independent of the flow, and need not be considered in this general explanation.

XLIII. *A Study of an Apparatus for the Determination of the Rate of Diffusion of Solids dissolved in Liquids. By* Albert Griffiths, *M.Sc.(Vic.), A.R.C.S.(Lond.).**

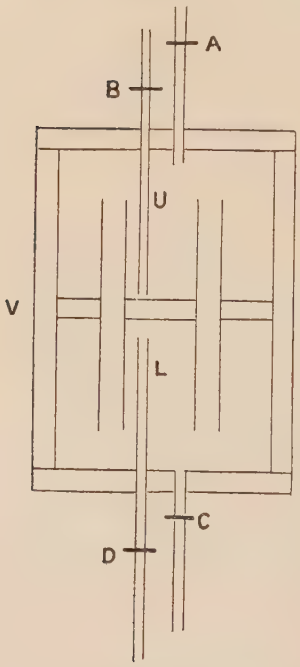
Section I.

For some time the author has been engaged in some experiments with the object of determining the coefficient of diffusion of bodies dissolved in water. It may be some years before experimental results of any decided value are obtained; and perhaps he will be allowed to give an account of the calculations involved, and of the methods he has adopted in obtaining an idea of the probable order of magnitude of the errors to which the apparatus is liable.

* Read March 10, 1899.

The apparatus consists of a vessel V, divided into two compartments, U and L, by a diaphragm through which pass a number of equal vertical tubes of which two only are shown in the figure. Two tubes, A, B, provided with stop-cocks, pass into U. One, A, just enters U; the other passes down to the bottom. Similarly two tubes C and D pass into L; C just enters, and D reaches to the top. The vessel is first completely filled with water, and in the case of a substance which produces an aqueous solution with a specific gravity greater than unity, the tubes A and B are closed, and the solution is passed in through C, the water of L being allowed to escape through D. Diffusion along the vertical tubes now commences, and the compartments U and L are alternately and periodically refilled with pure water and solution respectively. A quantitative analysis is made of the liquid taken from the upper compartment.

Fig. 1.



Section II.—*Calculation of the Coefficient of Diffusion.*

Lord Kelvin has solved the problem of the flow of electricity along a cable possessing appreciable capacity. This solution can readily be transferred to the problem under consideration.

It will be assumed that the upper extremities of the tubes are kept in contact with pure water, and the lower extremities in contact with a solution of constant strength.

Let γ = quantity of dissolved substance which enters the upper compartment in unit time, when the

combined *sectional area* of the tubes equals *unity*.

k = coefficient of diffusion (C.G.S. system), assumed to be a constant.

L = length of tubes in centims.

T = quantity of substance per c. c. in the lower compartment.

t = time in seconds.

Then

$$\gamma = \frac{k\mathrm{T}}{\mathrm{L}} \left\{ 1 + 2 \sum_{i=1}^{i=\infty} (-1)^i e^{\frac{-i^2\pi^2 kt}{\mathrm{L}^2}} \right\},$$

where i is any integer.

[If $\frac{\mathrm{L}^2}{\pi^2 k} \log \frac{4}{3}$ be taken as the unit of time, the topmost of the series of curves given on page 72, vol. ii., of Kelvin's 'Mathematical and Physical Papers,' shows graphically how γ rises to a maximum.]

Let q = total quantity of substance transmitted in t seconds. Integrating the preceding expression with respect to t, we obtain

$$q = \frac{k\mathrm{T}}{\mathrm{L}} \left\{ t + 2 \sum_{i=1}^{i=\infty} (-1)^i \frac{\mathrm{L}^2}{i^2\pi^2 k} - 2 \sum_{i=1}^{i=\infty} (-1)^i \frac{\mathrm{L}^2}{i^2\pi^2 k} e^{\frac{-i^2\pi^2 kt}{\mathrm{L}^2}} \right\} \quad (1)$$

When t *is very large* the third term within the brackets may be neglected, and

$$q = \frac{k\mathrm{T}t}{\mathrm{L}} + \frac{2\mathrm{LT}}{\pi^2} \sum_{i=1}^{i=\infty} \frac{(-1)^i}{i^2} \quad . \quad . \quad . \quad . \quad (2)$$

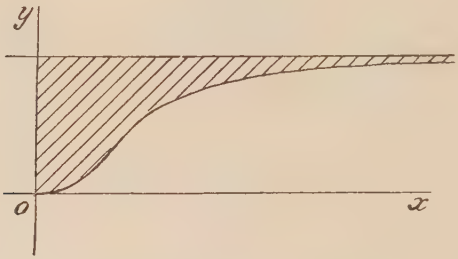
or

$$q = \frac{k\mathrm{T}t}{\mathrm{L}} - \frac{2\mathrm{LT}}{\pi^2} \frac{\pi^2}{12} = \frac{k\mathrm{T}t}{\mathrm{L}} - \frac{\mathrm{LT}}{6}. \quad . \quad . \quad . \quad (3)$$

The last equation shows that if t is large the quantity transmitted equals $k\mathrm{T}t/\mathrm{L}$ minus a quantity which is independent of the coefficient of diffusion. Thus if in the adjoining figure abscissæ represent times, and ordinates the quantities

diffused per second, the shaded area between the asymptote and the curve is independent of the coefficient of diffusion.

Fig. 2.



The table below gives an idea of the rate at which the steady state is attained when $L=4{\cdot}043$, and $k=3\times10^{-6}$. The unit of quantity is the amount transmitted per week on the attainment of the steady state.

No. of Weeks.	Total Quantity.	Quantity per Week.
1	0·104	0·104
2	0·703	0·599
3	1·567	0·864
4	2·519	0·952
5	3·505	0·986
6	4·500	0·995
7	5·498	
8	6·498	
9	7·498	
10	8·498	

[The above table gives the *integral* quantities transmitted. The curve already mentioned shows the way in which the rate of transmission varies. The unit abscissa $\frac{L^2}{\pi^2 k}\log_e\frac{4}{3}=$ ·2626 of a week.]

SECTION III.—*Error due to Differences of Temperature.*

The object of this section is to obtain the *order of magnitude* of the errors produced by differences in temperature between the various tubes.

For simplicity, two tubes only will be considered whose lengths and sectional areas are the same. The expansion of the material of the tubes will be neglected. To make the problem amenable to mathematical treatment, it will be assumed that the density of water equals $(1-\alpha\theta)$, where θ is the temperature, and α is approximately equal to an ideal coefficient of expansion. It will also be assumed that

$$d=(1-\alpha\theta+t),$$

where d = the density of the solution,

t = the concentration.

Let θ_1 = the temperature of one tube.
v_1 = the velocity of the liquid *up* the tube.
θ_2 = the temperature of the second tube.
v_2 = the velocity of the liquid *up* the tube.
L = the length of each tube, assumed constant.
k = the coefficent of diffusion, assumed constant.
T = the concentration of the solution at the bottom of each tube.

From considerations similar to those adopted in "Diffusive Convection"*, Sect. II. and III., it can be shown that, neglecting viscosity,

$$g(1-\alpha\theta_1)\mathrm{L}+\frac{g\mathrm{TL}}{1-e^{-\frac{v_1\mathrm{L}}{k}}}-\frac{kg\mathrm{T}}{v_1}$$
$$=g(1-\alpha\theta_2)\mathrm{L}+\frac{g\mathrm{TL}}{1-e^{-\frac{v_2\mathrm{L}}{k}}}-\frac{kg\mathrm{T}}{v_2}.$$

Dividing throughout by g, and expanding,

$$(1-\alpha\theta_1)\mathrm{L}+\mathrm{TL}\left(\frac{k}{v_1\mathrm{L}}+\frac{1}{2}+\frac{v_1\mathrm{L}}{12k}+\&c.\right)-\frac{k\mathrm{T}}{v_1}$$
$$=(1-\alpha\theta_2)\mathrm{L}+\mathrm{TL}\left(\frac{k}{v_2\mathrm{L}}+\frac{1}{2}+\frac{1}{12}\frac{v_2\mathrm{L}}{12k}+\&c.\right)-\frac{k\mathrm{T}}{v_2};$$

* *Ante*, pp. 230–243; Phil. Mag. Nov. 1898, pp. 453–465.

and

$$(v_2-v_1)=\frac{12k\alpha(\theta_2-\theta_1)}{\mathrm{TL}},\ \text{approx.}$$

Now

$$\frac{v_1}{v_2}=-\left(\frac{1-\alpha\theta_2}{1-\alpha\theta_1}\right);$$

hence

$$v_2\left\{1+\frac{1-\alpha\theta_2}{1-\alpha\theta_1}\right\}=\frac{12k\alpha(\theta_2-\theta_1)}{\mathrm{TL}},$$

and

$$\frac{v_2\mathrm{L}}{k}=\frac{12\alpha(1-\alpha\theta_1)(\theta_2-\theta_1)}{\mathrm{T}(2-\alpha\theta_1+\theta_2)};$$

similarly

$$\frac{v_1\mathrm{L}}{k}=\frac{12\alpha(1-\alpha\theta_2)(\theta_1-\theta_2)}{\mathrm{T}(2-\alpha\theta_1+\theta_2)}.$$

The correcting factor equals

$$\frac{\left\{1+\frac{1}{2}\frac{v_1\mathrm{L}}{k}+\frac{1}{12}\frac{v_1\mathrm{L}^2}{k^2}\right\}+\left\{1+\frac{1}{2}\frac{v_2\mathrm{L}}{k}+\frac{1}{12}\frac{v_2^2\mathrm{L}^2}{k^2}\right\}}{2}$$

$$=1+\frac{1}{4}\left(\frac{v_1\mathrm{L}}{k}+\frac{v_2\mathrm{L}}{k}\right)+\frac{1}{24}\left(\frac{v_1^2\mathrm{L}^2}{k^2}+\frac{v_2^2\mathrm{L}^2}{k^2}\right)$$

$$=1+\frac{3}{2}\frac{\alpha^2(\theta_2-\theta_1)^2}{\mathrm{T}}+\frac{3\alpha^2(\theta_2-\theta_1)^2}{\mathrm{T}^2},\ \text{approx.}$$

As an example, if $(\theta_2-\theta_1)=0°{\cdot}1$ C., $\mathrm{T}=0{\cdot}1$, and $\alpha=0{\cdot}00015$, the correcting factor equals 1·00000007.

When T is a small fraction of unity, as in general will be the case, the correcting factor equals

$$1+\frac{3\alpha^2(\theta_2-\theta_1)^2}{\mathrm{T}^2},\ \text{approx.}$$

It may be noted that, approximately,

$$v_2=-v_1=\frac{12k\alpha(\theta_2-\theta_2)}{\mathrm{TL}}.$$

As an example, let $k=3\times10^{-6}$, $\alpha=0{\cdot}00015$, $(\theta_2-\theta_1)=0{\cdot}1$, $\mathrm{T}=0{\cdot}1$, $\mathrm{L}=4$ cm.

Then

$$v_2 = 1{\cdot}35 \times 10^{-9} \text{ cm. per second,}$$
$$= 0{\cdot}04 \text{ cm. per year.}$$

SECTION IV.—*Errors due to Changes of Temperature of Apparatus as a whole.*

Apart from the effects of any differences of temperature which may exist between the tubes themselves, there will in general be a flow set up along them one way or the other when the apparatus as a whole is heated.

If the upper and lower compartments were of equal volume and the coefficients of expansion and the compressibilities of water and solution were the same, there would be no movement along the tubes; the liquid would simply be compressed slightly. The actual motion could only be calculated from a knowledge of the various coefficients of expansion and the compressibilities. The author has not the data for an accurate calculation, but to give an idea of the order of magnitude of the error, the case will be considered in which the taps of the upper compartment are open.

Let V = volume of lower compartment.
α = coefficient of cubical expansion of liquid.
θ = rise in temperature per second.

It will be assumed that the temperature increases at a constant rate, and that the steady state has been attained.

Let v = velocity *up* the tubes per second.
L = length of each tube.

Then $$v = \frac{V\alpha\theta}{A}.$$

The quantity of dissolved substance transmitted per second (see Diffusive Convection, Sec. IV.) equals, when VL/k is small,

$$\frac{kAT}{L}\left(1 + \tfrac{1}{2}\frac{vL}{k}\right) \text{approx.,}$$

i. e. $$\frac{kAT}{L}\left(1 + \tfrac{1}{2}\frac{VL\alpha\theta}{Ak}\right).$$

As an example, let the rise of temperature be 2° C. per week; let $\alpha=0{\cdot}00015$, L=4 cm., V=40 c.c., A=0·4, and $k=3\times10^{-6}$. Then $VL\alpha\theta/2Ak$ equals 0·033; and the error due to the flow produced by the expansion is more than 3 per cent.

It may be mentioned that if a bubble of air is present in the lower compartment, it increases the flow, and the corresponding error, considerably.

Even if the upper taps are not left continuously open, there is an error corresponding to the above; for opening the top taps will allow a sudden rush to take place up the tubes.

It is obvious that a diminution of temperature produces an error in the opposite direction; and the error produced by a fluctuating temperature will probably not be considerable in the long run.

Section V.—*Errors due to Changes in Volume produced by the Weakening of the Solution through Diffusion.*

In general, when a solution loses some of the dissolved substance by diffusion it diminishes in volume. The diminution depends on the nature of the dissolved substance and varies with the concentration. The diminution for any given loss can be calculated from a table of densities. For example, an aqueous solution of copper sulphate at a temperature of 23°·3 containing 80 grams to the litre has a density of 1·080, whereas a solution containing 160 grams to the litre has a density of 1·154. It can be readily shown that the addition of 1 gram of anhydrous copper sulphate, between the given limits, on the average increases the volume by 0·075 c.c. approx.

As in Section IV. it will be assumed that the upper compartment is open to the atmosphere.

Since the diminution of volume of the lower compartment equals the quantity of copper sulphate transmitted multiplied by 0·075, with the aid of Sec. I. "Diffusive Convection," it can readily be seen that

$$v=\frac{-0{\cdot}075\times T\times v}{1-e^{-\frac{vL}{k}}}.$$

Hence, neglecting the second and higher powers of $\frac{vL}{k}$,

$$\frac{vL}{k} = 0{\cdot}075 \times T.$$

Let $T = 0{\cdot}12$, then $\frac{vL}{k} = 0{\cdot}009$.

The fractional diminution in the quantity transmitted equals $vL/2k$, i.e. 0·0045. Thus the motion along the tubes causes a diminution in the quantity transmitted by about one half per cent.

With salts whose solutions experience greater changes in volume, the error may be very appreciable.

Thus, between the limits of density 1·081 and 1·159, the addition of 1 gram of sodium chloride causes an average increase in volume of 0·35 c.c.; and with $T = 0{\cdot}12$, the error would be 2·1 per cent.

It may be noted that when a small quantity of copper sulphate is added to water, the resultant volume is about the same as the original volume of the water. Hence if the lower compartment were left open to the atmosphere instead of the upper one, there would be no error from the cause under consideration.

As before errors are produced when the taps are opened intermittently.

SECTION VI.—*Errors due to Circulations produced when the Liquids of the Upper and Lower Compartments are renewed.*

When the water in the upper compartment is renewed the pressure at the inlet is greater than the pressures at the outlet, and in consequence the pressures at the upper extremities of the tubes are not constant. Hence flows take place up certain tubes and down others.

Let there be n tubes; and let p_1, p_2, &c. be the respective *excesses* of the pressures at the tops of the tubes above the *average* pressure. Assuming the attainment of the steady state, and making the usual approximations, it can readily be shown that the correcting factor equals

$$1 + \frac{12}{g^2 T^2 L^2}\left(\frac{p_1^2 + \ldots p_n^2}{n}\right).$$

Let p_r be the greatest of the p's; then the fractional error is less than $12\,p_r^2/g^2T^2L^2$. Neglecting inertia effects, the value of p_r must be less than the difference of pressure between the inlet and outlet, an estimate of which could be made practically. The author has not made any practical measurements, but has made a guess as to the order of magnitude of the difference of pressure by making a crude analogy between the upper compartment and a cylindrical tube. Thus, if 100 c.c. of water pass along a tube 3 cm. in diameter in an hour, the difference of pressure (assuming the formula for a capillary to hold) between two points on the axis 3 centim. apart, if we take the coefficient of viscosity equal to 0·01, is 0·004 dyne. Let $p_r=0{\cdot}01$, $T=0{\cdot}1$, $L=4$, then the fractional error equals

$$12\times(\cdot01)^2/(981)^2(0{\cdot}1)^2\times4^2, \text{ or } 7{\cdot}8\times10^{-9}.$$

It is clear that the error from this cause is not appreciable.

Section VII.—*Error due to Diffusive Convection.*

Inequalities in the lengths of the tubes and inaccuracies in placing them in the diaphragm will, in general, produce convection-currents. There is no intrinsic difficulty in calculating flows of small magnitude, if the tubes are of unequal length; but, to simplify the algebra, in the following the tubes will be taken of equal length and of equal sectional area.

Let n = number of tubes.

δ_1 = distance between the top of one tube and the *average* of the tops of all the tubes.

δ_r = corresponding distance for the rth tube.

v_1 = velocity up the first tube.

v_r = „ „ rth tube.

Making the usual assumptions, it can be shown that

$$\frac{(v_1-v_r)L^2}{12k}=\delta_r-\delta_1.$$

Noting that $\Sigma v_r=0$, $\Sigma\delta_r=0$, it can readily be shown that

$$\frac{v_rL}{k}=\frac{-12\delta_r}{L},$$

and that the correcting factor equals

$$1 + \frac{12}{L^2}\frac{\delta_1^2 + \ldots \delta_n^2}{n}.$$

With ordinary care in the construction of the apparatus, the correcting factor will deviate very little from unity.

Section VIII.—*Some Experimental Results.*

One apparatus has 15 tubes, of average length 4·043 centim., and a total sectional area 0·4102 square centim. It was placed in a cellar the temperature of which fluctuated through a range of about 3° C. per week. The lower compartment contained a solution of copper sulphate holding 0·0506 of copper to the c.c. At intervals of a week the upper compartment was renewed with pure water ; similarly the lower compartment was renewed with fresh solution. The tubes were originally full of pure water. The average temperature was about 8° C. By the end of the first week 0·0026 gram of copper had been transmitted ; by the end of the second a *total* of 0·0077 ; end of third, 0·0144 ; end of fourth, 0·0226 ; end of fifth, 0·0311 ; end of sixth, 0·0403 ; end of eighth, 0·0559 ; end of ninth, 0·0662.

Taking as a basis the quantity transmitted in nine weeks, the coefficient of diffusion equals $28{\cdot}7 \times 10^{-7}$. This result, and others not recorded, indicate that the errors due to the various causes studied are, at any rate, not enormous.

To diminish the variations of temperature, in one experiment the whole apparatus was put inside a copper cylinder; but, unfortunately, some air-bubbles were accidentally allowed to form within the compartments, and the quantities of copper sulphate transmitted were irregular and of large value. At present the author is experimenting with the tubes full of a weak gelatine jelly, with the intention, ultimately, of stopping each tube with a short plug of jelly, and thus preventing circulations of all sorts. The plugs will probably be made insoluble with formaldehyde. Mr. Hibbert has helped me with the analyses, and the results are hopeful ; but, as already indicated, progress is inevitably slow.

University College, Sheffield.

Discussion.

The President described an apparatus he had used for a thermostat. A double-walled cylinder of copper sheet, with a little water in the interspace, is exhausted at atmospheric temperature until the water boils; it is then sealed up. Water-vapour is a powerful equaliser of temperature, and a vapour-jacket of this kind is very efficient when it is required to maintain uniform temperature—not constant temperature.

Mr. Watson described an arrangement used by Mr. E. H. Griffiths in which tap-water is employed as a negative source and a gas-flame as a positive source—with extremely good results—as a thermostat for constant and for uniform temperatures of about 15° C.

XLIV. *The Joule-Thomson Thermal Effect; its Connexion with the Characteristic Equation, and some of its Thermodynamical Consequences.* *By* E. F. J. Love, *M.A., F.R.A.S., Assistant Lecturer and Demonstrator of Natural Philosophy in the University of Melbourne**.

(1) *Introductory.*

The results obtained by Lord Kelvin and the late Dr. Joule in their famous investigation into the "Thermal Effects of Fluids in Motion" have hitherto been utilized almost exclusively for one special purpose, viz. for determining the relation between various gas-thermometer scales and the absolute scale of temperature. But this is not the only information that can be obtained from them; and the present paper has been written partly in order to deduce some of their further consequences.

This deduction, however, is not the writer's sole object. He desires to indicate the relation which must exist between the formula assigned to the Joule-Thomson effect, considered as a function of the temperature, and the particular form adopted for the characteristic equation of a gas: in this way it is

* Read February 24, 1899.

possible to supply some sort of theoretical basis to the various formulæ put forward by different investigators, and to obtain some idea as to which of them is the most likely to be correct. With a view to an orderly development of the subject this discussion will be taken first; it will be followed by a deduction, from Joule and Lord Kelvin's results, of the relation between the intrinsic energy of a gas and its volume, upon which is based a new method of calculating the ratio of the two principal specific heats of a gas. A concluding paragraph deals with some points in the thermodynamics of substances at their temperature of maximum density.

(2) *Notation.*

The following scheme of notation is adhered to in this paper:—

Pressure of a substance	p
Volume of unit mass	v
Absolute temperature	T
Density of substance	ρ
Specific heat at constant pressure	K_p
Specific heat at constant volume	K_v
Intrinsic energy of unit mass	U
Joule-Thomson cooling effect per atmosphere rise of pressure	θ
Standard pressure of 1 atmosphere	Π
Mechanical equivalent of heat	J
Coefficient of dilatation	σ
Quantity of heat	Q

The letters A, B, C, R, a, b, α, β, γ, δ denote constants occurring in various equations; subscribed letters indicate that the quantities denoted by them are supposed constant.

(3) *The Joule-Thomson Effect and the Gas-Equation.*

The reasoning of this section is based throughout on Lord Kelvin's well-known equation

$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p - v = -\mathrm{J}\,.\,\mathrm{K}_p \frac{d\mathrm{T}}{dp},$$

where dT denotes the rise of temperature in a "porous plug" experiment corresponding to a rise of pressure dp on passing the plug.

(1) If we adopt the ordinary Boyle-Charles equation of a perfect gas

$$pv=\mathrm{RT},$$

then we can show at once by differentiation that

$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p - v = 0;$$

in other words, the Joule-Thomson thermal effect is zero, a result which the experiments conclusively disprove.

(2) A very different result is obtained if we adopt van der Waals's equation; as we shall see, it leads directly to the expression recently* formulated by Rose-Innes. We start from the equation

$$\left(p+\frac{a}{v^2}\right)(v-b)=\mathrm{RT};$$

dividing throughout by $v-b$, and differentiating with respect to T, on the assumption $p=$const., we obtain

$$-\frac{2a}{v^3}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p = \frac{\mathrm{R}}{v-b} - \frac{\mathrm{RT}}{(v-b)^2}\cdot\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p;$$

$$\therefore \left(\frac{\partial v}{\partial \mathrm{T}}\right)_p = \frac{\dfrac{\mathrm{R}}{v-b}}{\dfrac{\mathrm{RT}}{(v-b)^2}-\dfrac{2a}{v^3}},$$

i. e.

$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p = (v-b)\left\{1-\frac{2a}{\mathrm{RT}v}\left(1-\frac{b}{v}\right)^2\right\}^{-1}.$$

Multiplying out on the right-hand side and neglecting powers and products of a and b (since both are very small) we obtain

$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p = v-b+\frac{2a}{\mathrm{RT}}; \quad . \quad . \quad . \quad . \quad \text{(q. p.)}$$

whence

$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p - v = \frac{2a}{\mathrm{RT}} - b;$$

* Phil. Mag. March 1898.

i. e. $$-\mathrm{J}\,.\,\mathrm{K}_p\,.\frac{d\mathrm{T}}{dp}=\frac{2a}{\mathrm{RT}}-b.$$

Following Joule and Kelvin, let us write

$$-\frac{d\mathrm{T}}{dp}=\frac{\theta}{\Pi},$$

whence we obtain

$$\theta=\frac{\Pi}{\mathrm{J}\,.\,\mathrm{K}_p}\left(\frac{2a}{\mathrm{RT}}-b\right),$$

$$=\frac{\alpha}{\mathrm{T}}-\beta,$$

where α and β are constants. This is exactly the formula proposed by Rose-Innes, from an examination of the results of the experiments of Joule and Lord Kelvin; it is not without interest to find that this formula is directly deducible from van der Waals's equation.

(3) The result obtained in the foregoing paragraph suggested the advisability of making a similar calculation on the basis of Clausius's characteristic equation

$$\left\{p+\frac{\mathrm{A}}{\mathrm{T}(v+\mathrm{C})^2}\right\}(v-\mathrm{B})=\mathrm{RT},$$

which admittedly represents the behaviour of the easily condensible gases with greater accuracy than van der Waals's equation does.

Dividing throughout, as before, by $v-\mathrm{B}$, and differentiating with respect to T (assuming $p=$constant), we obtain

$$-\frac{\mathrm{A}}{\mathrm{T}^2(v+\mathrm{C})^2}-\frac{2\mathrm{A}}{\mathrm{T}(v+\mathrm{C})^3}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p=\frac{\mathrm{R}}{v-\mathrm{B}}-\frac{\mathrm{RT}}{(v-\mathrm{B})^2}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p;$$

i. e.

$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p=\frac{\dfrac{\mathrm{R}}{v-\mathrm{B}}+\dfrac{\mathrm{A}}{\mathrm{T}^2(v+\mathrm{C})^2}}{\dfrac{\mathrm{R}}{(v-\mathrm{B})^2}-\dfrac{2\mathrm{A}}{\mathrm{T}^2(v+\mathrm{C})^3}}$$

$$=\left\{v-\mathrm{B}+\frac{\mathrm{A}}{\mathrm{RT}^2}\left(\frac{v-\mathrm{B}}{v+\mathrm{C}}\right)^2\right\}\left\{1-\frac{2\mathrm{A}}{\mathrm{RT}^2}\cdot\frac{(v-\mathrm{B})^2}{(v+\mathrm{C})^3}\right\}^{-1}.$$

Neglecting powers and products of the small quantities A, B, and C, this becomes

$$\mathrm{T}\left(\frac{\partial v}{\partial \mathrm{T}}\right)_p-v=\frac{3\mathrm{A}}{\mathrm{RT}^2}-\mathrm{B};\quad\ldots\quad(\text{q. p.})$$

whence (by the same reasoning as in the previous discussion) we obtain

$$\theta = \frac{\gamma}{T^2} - \delta,$$

where γ and δ are constants—an expression which differs from that given by Rose-Innes in that it involves the square of the temperature instead of its first power.

It may not be uninteresting to compare the results furnished by this formula with the figures published by Rose-Innes; the comparison is here given for CO_2 only, seeing that the behaviour of air and hydrogen is expressed about as well by van der Waals's equation as by that of Clausius, and so the calculation in their case is hardly worth making.

Plotting the values of θ and $\frac{1}{T^2}$ for CO_2 as ordinates and abscissæ we obtain a remarkably good straight line; from it the following values were deduced,

$$\gamma = 4{\cdot}11 \times 10^5; \qquad \delta = 0{\cdot}88.$$

With these constants the values of θ were recalculated for the experimental temperatures, and the annexed table gives the results. It will be seen that, slight as are the differences between the results of observation and those furnished by Rose-Innes's formula, the differences in the case of the new formula are still smaller, as would naturally be anticipated from the above discussion.

TABLE.

Temp. Centigrade.	θ (observed).	θ (calculated) (Rose-Innes).	Differences (obs.—calc.).	θ (calculated) (new formula).	Differences (obs.—calc.).
0·0°	4·64	4·60	+0·04	4·64	0·00
7·4	4·37	4·35	+0·02	4·35	+0·02
35·6	3·41	3·49	−0·08	3·43	−0·02
54·0	2·95	3·02	−0·07	2·96	−0·01
93·5	2·16	2·16	0·00	2·18	−0·02
97·6	2·14	2·08	+0·06	2·12	+0·02

Several remarks are suggested by the discussion just given. In the first place, we observe that the Joule-Kelvin formula for the thermal effect would hold good for gases whose co-volume was zero, or negligible in comparison with the "molecular pressure"; it is not unlikely, indeed, that if the original investigators had plotted θ and $\frac{1}{T^2}$, instead of $\frac{\theta}{T^2}$ and T, they would have arrived at the formula here suggested. Be that as it may, their suggestion that the temperature enters into the expression in its second power is probably very near the truth.

Another point worth noticing is the testimony, borne by the calculations, to the remarkable accuracy of these experiments; this statement holds good whether Rose-Innes's formula or the present writer's be adopted, though most markedly, perhaps, in the latter case. This remark seems worth making as, in the present writer's opinion, too little credit has in the past been attached to the experiments in this particular; for example, Lehfeldt's statement*, that we probably do not know the value of θ for any single case within 10 per cent., seems to be a good deal wide of the mark.

Our discussion further supplies an answer to the critique on Clausius's formula expressed by Natanson† in his able memoir on the Joule-Thomson effect. He affirms that the results of the experiments render the existence of any physical quantity corresponding to Clausius's β—styled C in this paper—improbable. As a matter of fact, the experiments give no evidence at all on this point; for the disputed constant enters only into terms which involve the second order of small quantities, terms which would be insensible in any "porous plug" experiments hitherto performed.

(4) *Some Thermodynamical Consequences of Joule and Thomson's Experiments.*

(a) *The Relation between the Intrinsic Energy of ordinary Gases and their Volume.*—If a substance—whether gaseous or not—be allowed to expand infinitesimally without doing

* Phil. Mag. April 1898. † Wied. *Ann.* xxxi. p. 502.

work or gaining heat, as in a "porous plug" experiment, we know that

$$d(\mathrm{U}+pv)=0\ ;$$

if at the same time its temperature falls through a range $d\mathrm{T}$, we must communicate to it an amount of heat (reckoned in ergs) $=\mathrm{J}\,.\,\mathrm{K}_v\,.\,d\mathrm{T}$ if we desire to restore the substance to its initial temperature without gaining or expending work.

Hence any infinitesimal isothermal change in the intrinsic energy may be calculated as follows:—

1st. Let the substance expand through a porous plug till its final volume is attained; the gain of internal energy is $-(pdv+vdp)$ and the fall of temperature $d\mathrm{T}$.

2nd. Heat the substance at constant volume till its initial temperature is regained; this increases the intrinsic energy by an amount $\mathrm{J}\,.\,\mathrm{K}_v\,.\,d\mathrm{T}$. Hence

$$d_{\mathrm{T}}\mathrm{U}=-pdv-vdp+\mathrm{J}\,.\,\mathrm{K}_v\,.\,d\mathrm{T}.$$

This is true for all substances; to apply it to gases we proceed as follows:—

Writing

$$d\mathrm{T}=-\frac{\theta}{\Pi}dp,$$

as is done by Joule and Lord Kelvin, we obtain

$$d_{\mathrm{T}}\mathrm{U}=-pdv-\left(v+\frac{\theta}{\Pi}\,.\,\mathrm{J}\,.\,\mathrm{K}_v\right)dp\,;$$

whence

$$\left(\frac{\partial\mathrm{U}}{\partial v}\right)_{\mathrm{T}}=-\left\{p+\left(v+\frac{\theta}{\Pi}\,.\,\mathrm{J}\,.\,\mathrm{K}_v\right)\frac{dp}{dv}\right\},$$

where it is to be noted that $\frac{dp}{dv}$ is not a *partial* differential coefficient.

To evaluate this quantity we must determine $\frac{dp}{dv}$; and to do this we must assume some form for the gas-equation. It does not much matter which we take—as we shall see presently—for an approximation is all that is required; anything further is rendered needless by the fact that $\left(\frac{\partial\mathrm{U}}{\partial v}\right)_{\mathrm{T}}$ is always small compared with $\left(\frac{\partial\mathrm{U}}{\partial\mathrm{T}}\right)_v$.

We will employ van der Waals's formula ; then

$$p = \frac{RT}{v-b} - \frac{a}{v^2};$$

$$\therefore \frac{dp}{dv} = \frac{R}{v-b}\frac{dT}{dv} - \frac{RT}{(v-b)^2} + \frac{2a}{v^3};$$

but $$\frac{dT}{dv} = -\frac{\theta}{\Pi} \cdot \frac{dp}{dv}, \quad \text{and} \quad \frac{RT}{v-b} = p + \frac{a}{v^2},$$

therefore

$$\frac{dp}{dv}\left(1 + \frac{\theta}{\Pi}\frac{R}{v-b}\right) = \frac{2a}{v^3} - \frac{p + \frac{a}{v^2}}{v-b},$$

i. e.

$$\frac{dp}{dv} = \left\{\frac{2a}{v^3} - \frac{p + \frac{a}{v^2}}{v-b}\right\}\left\{1 + \frac{\theta}{\Pi}\frac{R}{v-b}\right\}^{-1}.$$

Substituting this value of $\frac{dp}{dv}$ in the equation

$$\left(\frac{\partial U}{\partial v}\right)_T = -\left\{p + \left(v + \frac{\theta}{\Pi} \cdot J \cdot K_v\right)\frac{dp}{dv}\right\}$$

we obtain

$$\left(\frac{\partial U}{\partial v}\right)_T = \left\{v + \frac{\theta}{\Pi} \cdot J \cdot K_v\right\} \frac{p + \frac{a}{v^2} - \frac{2a}{v^3}}{1 - \frac{\theta}{\Pi}\frac{R}{v-b}} - p.$$

Since $\frac{a}{v^2}$ is in general small compared with p, and b and $\frac{R\theta}{\Pi}$ are both small compared with v, we have

$$\left(\frac{\partial U}{\partial v}\right)_T = \left(v + \frac{\theta}{\Pi} \cdot J \cdot K_v\right)\frac{p}{v} - p \quad . \quad . \quad \text{(q. p.)}$$

$$= \frac{\theta}{\Pi} \cdot \frac{p}{v} \cdot J \cdot K_v$$

$$= \frac{\theta}{\Pi} \cdot p \cdot \rho \cdot J \cdot K_v.$$

We should have obtained the same expression if we had employed either the Boyle-Charles equation or that of

Clausius ; in the former case as the direct result, in the latter as an approximation of the same order as that made above.

We may now evaluate this quantity numerically for air at standard pressure and temperature ; for this purpose we have the approximate values $\theta=0{\cdot}92$; $p=\Pi$; $\rho=0{\cdot}00129$; $K_v=0{\cdot}17$; $J=4{\cdot}2\times10^7$; whence

$$\left(\frac{\partial U}{\partial v}\right)_T=8{\cdot}5\times10^3. \quad . \quad . \quad . \quad . \quad \text{(q. p.)}$$

(b) *Ratio of the two Specific Heats of a Gas.*—We have for all bodies

$$K_v=\frac{1}{J}\left(\frac{\partial U}{\partial T}\right)_v; \quad K_p=\frac{1}{J}\left\{\left(\frac{\partial U}{\partial T}\right)_v+\left(\frac{\partial U}{\partial v}\right)_T\left(\frac{\partial v}{\partial T}\right)_p+p\left(\frac{\partial v}{\partial T}\right)_p\right\}$$

$$=K_v+\frac{\sigma}{J\rho}\left\{\left(\frac{\partial U}{\partial v}\right)_T+p\right\};$$

$$\frac{K_p}{K_v}=\frac{K_p}{K_p-\frac{\sigma}{J\rho}\left\{\left(\frac{\partial U}{\partial v}\right)_T+p\right\}}.$$

If, therefore, we make use of the value of $\left(\frac{\partial U}{\partial v}\right)_T$ as determined in the preceding subsection, we can determine this ratio for ordinary gases.

We give the result for air at standard pressure and temperature ; here $K_p=0{\cdot}2375$; $\sigma=0{\cdot}003665$; $\rho=0{\cdot}001293$; $J=4{\cdot}212\times10^7$*; $p=1{\cdot}0136\times10^6$; $\left(\frac{\partial U}{\partial v}\right)_T=8{\cdot}5\times10^3$; whence

$$\frac{K_p}{K_v}=1{\cdot}408.$$

This is exactly the value deduced from the velocity of sound ; it is also the mean of the results given by Röntgen (1·406), and by Jamin and Richard (1·41), who worked by totally different methods.

Had air been a perfect gas, $\left(\frac{\partial U}{\partial v}\right)_T$ would have been zero, and our formula would become

$$\frac{K_p}{K_v}=\frac{K_p}{K_p-p\cdot\frac{\sigma}{J\rho}}.$$

* Obtained by extrapolation from Rowland's results.

In this case the same values of K_p, p, σ, and ρ would give $\frac{K_p}{K_v} = 1{\cdot}400$, which is the value assigned by the kinetic theory for a perfect gas consisting of diatomic molecules. The Joule-Thomson effect, then, completely accounts for the difference between the "observed" and "theoretical" values of this important ratio; and verifies the suggestion, made long ago, that the "imperfection" of the gas is probably the source of this difference.

(5) *The Maximum Density Point.*

Some interesting thermodynamic properties of substances at their temperature of maximum density are deducible from the equation, employed in the last section,

$$K_p - K_v = \frac{\sigma}{J\rho}\left\{\left(\frac{\partial U}{\partial v}\right)_T + p\right\};$$

as the present writer does not remember to have seen them stated anywhere, he ventures to insert them here. Lord Kelvin * has shown that the following equation holds good for all substances,

$$\frac{K_p}{K_v} = \frac{K_p}{K_p - T \cdot \frac{\sigma^2}{J\rho} \cdot e_T}$$

(where e_T denotes the isothermal elasticity, and the notation is changed to that used in this paper); hence we get

$$K_p - K_v = T \cdot \frac{\sigma^2}{J\rho} \cdot e_T;$$

combining this equation with the previous one we obtain

$$\left(\frac{\partial U}{\partial v}\right)_T = T \cdot \sigma \cdot e_T - p$$

for all substances.

If, then, a substance have a maximum density at any particular temperature, σ vanishes for that temperature, and we get

$$\left(\frac{\partial U}{\partial v}\right)_T = -p; \quad K_p = K_v.$$

But this is not all. The first law of thermodynamics,

* *Encyc. Brit.* art. "Elasticity."

written in its ordinary form, is

$$J \cdot dQ = \left(\frac{\partial U}{\partial T}\right)_v \cdot dT + \left\{\left(\frac{\partial U}{\partial v}\right)_T + p\right\} \cdot dv;$$

at the maximum density point $\left(\frac{\partial U}{\partial v}\right)_T + p$ vanishes, as we have just seen; consequently

$$J \cdot dQ = \left(\frac{\partial U}{\partial T}\right)_v \cdot dT.$$

If dT be zero dQ vanishes; in other words, *no heat is absorbed or evolved in any isothermal transformation at this temperature; i. e.* the latent heat of isothermal transformation is zero. Similarly dT vanishes if dQ be zero.

Many interesting conclusions may be drawn from this remarkable result; among them are the following:—

1st. The Joule-Thomson effect is zero for every substance at its maximum density, just as it is for an ideal perfect gas, though for a very different reason.

2nd. The *infinite number* of specific heats which every substance possesses is reduced at this point to *one.*

University of Melbourne,
December 5th, 1898.

Discussion.

Mr. Rose-Innes congratulated the author on having written an interesting paper on a difficult subject. At the same time he felt bound to acknowledge that he was out of sympathy with the general idea contained in the paper. The experimental difficulties that occurred in carrying out the Joule-Thomson measurements were so enormous that it was better to rely on them as little as possible, notwithstanding the great skill of the experimenters. The Joule-Thomson results could not be disregarded altogether, since they were necessary for the establishment of the thermo-dynamic scale; but once that scale had been set up, it was better to have recourse as much as possible to such experiments as those of M. Amagat on the compressibility of gases. He also pointed out that one of the deductions given in the paper from van der Waals's formula had already been given by van der Waals himself.

XLV. *On the Criterion for the Oscillatory Discharge of a Condenser.* *By* E. H. Barton, *D.Sc.*, *F.R.S.E.*, *and* W. B. Morton, *M.A.**

The ordinary condition for the oscillatory discharge of a condenser of capacity C through a wire of resistance R and inductance L, namely

$$C < \frac{4L}{R^2},$$

is obtained from the differential equation

$$0 = \frac{Q}{C} + R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

by making Q proportional to e^{xt} and expressing the condition that the resulting quadratic has imaginary roots. Now if account be taken of the distribution of the current in the wire, the well-known work of Maxwell and Lord Rayleigh † shows that the above differential equation must, for a straight wire, be supplemented by terms on the right,

$$-\tfrac{1}{12}R\alpha^2\mu^2\frac{d^3Q}{dt^3} + \tfrac{1}{48}R\alpha^3\mu^3\frac{d^4Q}{dt^4} - \tfrac{1}{180}R\alpha^4\mu^4\frac{d^5Q}{dt^5}, \text{ \&c., . . } \quad (2)$$

where α is the length of the wire divided by its resistance $\left(\frac{l}{R}\right)$, and μ the permeability of the material of the wire. For a curved wire we should probably have an equation of the same *form* with different coefficients. The question is, how do these additional terms affect the condition for oscillatory discharge?

The added terms have coefficients which are in general small compared with R. Lord Rayleigh (*loc. cit.*) takes, for iron, the value 10^4 for resistivity and 300 for μ. This would give for a wire of radius a, $\mu\alpha = \frac{3\pi a^2}{100}$; so that the coefficient of $\frac{d^3Q}{dt^3}$ would be, even for thick wire, less than $R \times 10^{-5}$. For copper the value would be still less. If, therefore, we

* Read March 24, 1899.

† Maxwell, 'Treatise,' vol. ii. § 690; Rayleigh, Phil. Mag. vol. xxi. p. 381 (1886).

put e^{xt} for Q, we have an algebraic equation of which the terms above the second are of small and decreasing importance. The effect of these small terms will be (1) to introduce very large roots corresponding to very rapid oscillations; these will clearly be of small amplitude, and will not affect the main phenomena of discharge: (2) to modify the original roots of the uncorrected quadratic equation. The cases of oscillatory and non-oscillatory discharge are separated by the case of equality of these displaced roots.

It is easy to see that the effect of the added terms will be to make the critical value on the simple theory $\left(C=\frac{4L}{R^2}\right)$ correspond actually to an oscillatory discharge. For in this case the graph of $y=\frac{1}{C}+Rx+Lx^2$ evidently touches the axis of x at a point on the negative side of the origin, viz. $x=-\frac{R}{2L}$, and lies entirely above the axis. If we compound with this the graph of the additional terms,

$$y=-\tfrac{1}{12}R\alpha^2\mu^2x^3 \text{ \&c.},$$

which, for negative x, also lies above the axis, we get the parabolic graph displaced upwards, so that the roots of the modified equation become imaginary.

To find to any desired degree of approximation the condition for equality of the chief roots of the equation, we may use the principle that a repeated root of an equation $f(x)=0$ is a root also of the derived equation $f'(x)=0$.

In what follows we shall write the differential equation of discharge in the form

$$0=\frac{Q}{C}+R\frac{dQ}{dt}+L\frac{d^2Q}{dt^2}+\phi\left(\frac{d}{dt}\right)Q, \quad . \quad . \quad . \quad (3)$$

and the corresponding algebraic equation

$$0=\frac{1}{C}+Rx+Lx^2+\phi(x), \quad . \quad . \quad . \quad . \quad (4)$$

using ϕ for the series of small terms in $\alpha^2\mu^2$, &c.

The derived equation is

$$0=R+2Lx+\phi'(x). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let the common root of these equations be $x=-\frac{R}{2L}+\theta$, where θ is small; then we can find θ to any degree of approximation from (5), which becomes

$$0=2L\theta+\phi'\left(-\frac{R}{2L}+\theta\right)$$
$$=2L\theta+\phi_0'+\phi_0''.\theta+\tfrac{1}{2}\phi_0'''.\theta^2+\&c.,. \quad . \quad . \quad (6)$$

where ϕ_0 is put for $\phi\left(-\frac{R}{2L}\right)$.

This gives

$$\theta=-\frac{\phi_0'}{2L}+\frac{\phi_0'\phi_0''}{4L^2}-\frac{\phi_0'}{16L^3}(2\phi_0''^2+\phi_0'\phi_0'''). \quad . \quad (7)$$

Equation (4) becomes, in terms of θ,

$$0=\frac{1}{C}-\frac{R^2}{4L}+L\theta^2+\phi\left(-\frac{R}{2L}+\theta\right). \quad . \quad . \quad . \quad (8)$$

Putting in this the value (7), we get the required condition in the form

$$\frac{1}{C}=\frac{R^2}{4L}-\phi_0+\frac{\phi_0''^2}{4L}-\frac{\phi_0'^2\phi_0''}{8L^2}, \quad . \quad . \quad . \quad . \quad (9)$$

which goes to the third order in ϕ_0 or the sixth in $\alpha\mu$.

Inserting the numerical values and arranging in powers of $\alpha\mu$, it comes out

$$\frac{1}{C}=\frac{R^2}{4L}-\frac{1}{96}\frac{R^4\alpha^2\mu^2}{L^3}-\frac{1}{768}\frac{R^5\alpha^3\mu^3}{L^4}+\frac{37}{46080}\frac{R^6\alpha^4\mu^4}{L^5}. \quad . \quad (10)$$

Since $\alpha=\frac{l}{R}$, where l is the length of the wire, we have the correcting terms expressed as a series in $\frac{\mu l}{L}$ thus

$$\frac{1}{C}=\frac{R^2}{4L}\left\{1-\frac{1}{24}\left(\frac{\mu l}{L}\right)^2-\frac{1}{192}\left(\frac{\mu l}{L}\right)^3+\frac{37}{11520}\left(\frac{\mu l}{L}\right)^4\ldots\right\}. \quad (11)$$

Equations (10), (11) show that the critical capacity is greater than that given by the simple theory.

It is interesting to compare with the above another and more physical way of treating the question. In a paper by one of us* read before the Physical Society on January 27th of this

* Barton, Proc. Phys. Soc. *ante*, p. 409.

year, expressions were obtained for the equivalent resistance and inductance of a wire for damped simple harmonic oscillations. The method of that paper was equivalent to putting for Q in equation (3) the value $e^{(ip-kp)t}$ and arranging the result in the form

$$0=\frac{1}{C}+(ip-kp)R''+(ip-kp)^2L''. \quad . \quad . \quad (12)$$

The real quantities R″, L″ gave the resistance and inductance required. To apply this to the present problem,—we can evidently get an approximation to the criterion sought by replacing R and L in the ordinary formula $C=\frac{4L}{R^2}$ by modified values appropriate to the case. These may be got approximately by putting, in the expressions for R″ and L″, $p=0$ and $kp=\frac{R}{2L}$.

The result obtained by this method agrees with (11) as far as the terms there given. Let us now examine what the process sketched above really amounts to as a mathematical treatment of equation (4), and to what order of small quantities its approximation holds good.

First we must put $ip-kp$ in ϕ and arrange in accordance with (12). Using Taylor's expansion, we have

$$\phi(ip-kp)=(ip-kp)\,.\,\frac{\phi(ip-kp)}{ip-kp}$$

$$=(ip-kp)\left[\frac{\phi(-kp)}{-kp}+ip\,.\left\{\frac{\phi'(-kp)}{-kp}-\frac{\phi(-kp)}{k^2p^2}\right\}+\ldots\right]$$

$$=(ip-kp)\left[-\frac{2\phi(-kp)}{kp}-\phi'(-kp)+\&c.\right]$$

$$+(ip-kp)^2\left[-\frac{\phi(-kp)}{k^2p^2}-\frac{\phi'(-kp)}{kp}+\&c.\right];$$

all terms in the square brackets, with the exception of those written, vanish with p.

Now put $p=0$, $kp=\frac{R}{2L}$, and referring to the complete equation (4) we see that the "equivalent" R″ and L″ become

respectively

$$R_0'' = R - \frac{4L}{R}\phi_0 - \phi_0',$$

$$L_0'' = L - \frac{4L^2}{R^2}\phi_0 - \frac{2L}{R}\phi_0'.$$

It may be easily verified that these expressions agree with those got from equations (15) and (16) of the paper quoted, when the values zero and $\frac{R}{2L}$ are put for p and kp respectively.

We see now that the use of the condition

$$C = \frac{4L_0''}{R_0''^2}$$

is equivalent, mathematically, to writing equation (4) in the form

$$0 = \frac{1}{C} + x\left\{R + \frac{2\phi(x)}{x} - \phi'(x)\right\} + x^2\left\{L - \frac{\phi(x)}{x^2} + \frac{\phi'(x)}{x}\right\},$$

putting for x inside the brackets the value $-\frac{R}{2L}$, and treating the equation as a quadratic with constant coefficients.

As regards the degree of approximation given by this method, if we expand $\frac{R_0''^2}{4L_0''}$, we get

$$\frac{1}{C} = \frac{R^2}{4L} - \phi_0 + \frac{\phi_0'^2}{4L} + \frac{\phi_0'^2}{2R^2}(2L\phi_0 + R\phi_0') \ldots,$$

which agrees with the accurate expansion in (9) up to and including terms of the *second* order in ϕ. Or expressing the condition in a series of powers of $\alpha\mu$, as in (10), the two series coincide up to and including terms in $\alpha^5\mu^5$.

March 2nd, 1899.

Discussion.

Prof. Lodge said that the result naturally to be expected of "throttling," viz., the increase of resistance, and decrease of self-induction, due to the current keeping to the outside of the conductor, would tend rather to damp out the oscillations than to favour them.

Prof. EVERETT observed that the equation was no longer a quadratic, and that the quadratic criterion as to whether the discharge was oscillatory or non-oscillatory did not hold. The paper appeared to be consistent with itself, and he considered that the authors had satisfactorily proved, in their discussion of the equation of current, that the effect of "throttling" was to increase the tendency towards the oscillatory mode of discharge.

Prof. LODGE admitted that the quadratic criterion did not hold; he thought it most likely that the authors, who evidently had gone into the matter with care, were right. At the same time he wished to call attention to the singular and unexpected character of their conclusion. If it turned out that it was correct, *i. e.*, that there was no slip in sign, it was a result upon which he would desire to congratulate them. The result for damped might easily be different from the result for maintained oscillations; but it was not clear what would happen at the exactly dead-beat condition.

XLVI. *Supplementary Note to Paper "On the Criterion for the Oscillatory Discharge of a Condenser"**. *By* Dr. E. H. BARTON *and* Prof. W. B. MORTON†.

IN the discussion which followed the reading of this paper before the Physical Society it was pointed out that the result obtained—viz., that on taking into account the distribution of the current in the wire a condenser having the critical capacity on the simple theory gives an oscillatory discharge—seems to be contradicted by the well-known fact that the resistance of a wire is greater, and its inductance less, for oscillatory than for steady currents. Both the increase of resistance and the decrease of inductance should favour a *non* oscillatory discharge. The explanation of the apparent paradox is to be found in the effect of the damping on the inductance. It was shown in the paper by one of us‡

* See page 465 *suprà*. † Read May 12, 1899.

‡ Barton, "The Equivalent Resistance and Inductance of a Wire to an Oscillatory Discharge:" read January 27th, 1899. Proc. Phys. Soc. *ante*, p. 409.

which has been already quoted, that the damping of the oscillations causes an increase in both R and L. When the damping is great and the frequency small, as in the neighbourhood of our critical case, what may be termed the *equivalent inductance* L″ becomes *greater* than the steady-current value. The investigation of the present paper shows that this increase of L outweighs the increase of R in its effect on the criterion for oscillatory discharge. It seems worth while to examine this effect of damping on the inductance a little more closely.

The formula for equivalent inductance for current of the form $e^{-kpt+ipt}$ is (eq. 16, *loc. cit.*)

$$L''=l\left[A+\mu\left(\frac{1}{2}+\frac{kp\alpha\mu}{6}-\frac{1-3k^2}{48}p^2\alpha^2\mu^2-\&c.\right)\right],$$

which shows that when the damping kp is sufficiently large compared with p, then $L''>l(A+\frac{1}{2}\mu)$, *i. e.* greater than the steady-current value. We can form an approximate notion of the stage at which this takes place by noticing that when $k=\frac{p\alpha\mu}{8}$ the small terms written above are all positive. If we take the numerical value for $\alpha\mu$ in case of iron, used in the beginning of the paper, viz. $\alpha\mu=\frac{3\pi a^2}{100}$, and put $a=1$ mm., we have roughly $\frac{1}{1000}$ as the value of $\alpha\mu$. This would make the necessary damping $k=\frac{p}{8000}$, or the logarithmic decrement per wave equal to $\frac{2\pi p}{8000}$ roughly.

Putting $p=2\pi n$, where n is the frequency, this would give as the ratio of one amplitude to the next of the same sign about $e^{\frac{n}{2000}}$.

Since the decrease of L with maintained oscillations is due to the concentration of current near the surface of the wire, it is at once suggested that we must have, in the present case, when the damping becomes important, an *axis*-concentration of current. The following investigation, by the method of Maxwell, shows that this is the case. In the discussion in the 'Treatise,' vol. ii. Art. 689, Maxwell expresses the current-density w at distance r from the axis of the wire by his

equation (3), which, modified by the introduction of μ (see Lord Rayleigh, Phil. Mag. May 1886), reads

$$-\pi\mu w = T_1 + 4T_2 r^2 + 9T_3 r^4 + \ldots + n^2 T_n r^{2n-2} + \ldots$$

The T's are functions of the time which are subsequently connected with each other by the equations (10) (again modified by the insertion of μ),

$$T_1 = \frac{\pi\mu}{\rho}\frac{dT}{dt}, \ldots T_n = \frac{\pi^n \mu^n}{\rho^n (n!)^2}\frac{d^n T}{dt^n},$$

where ρ is the resistivity.

T is then expressed in terms of the *total* current C by

$$\alpha\frac{dT}{dt} = -C + \tfrac{1}{2}\mu\alpha\frac{dC}{dt} - \tfrac{1}{6}\mu^2\alpha^2\frac{d^2C}{dt^2} + \&c.$$

The reduction of these equations enables us to express w in terms of C and its differentials with respect to the time, thus

$$\pi\alpha^2 w = C - \mu\alpha\frac{dC}{dt}\left(\frac{1}{2} - \frac{r^2}{a^2}\right) + \mu^2\alpha^2\frac{d^2C}{dt^2}\left(\frac{1}{6} - \frac{1}{2}\frac{r^2}{a^2} + \frac{1}{4}\frac{r^4}{a^4}\right), \&c.$$

To apply this to damped oscillations, put

$$C = e^{-kpt}\cos pt;$$

the right-hand side of the equation last written then becomes

$$e^{-kpt}\cos pt\left[1 + kp\alpha\mu\left(\frac{1}{2} - \frac{r^2}{a^2}\right) + \alpha^2\mu^2(k^2p^2 - p^2)\left(\frac{1}{6} - \frac{1}{2}\frac{r^2}{a^2} + \frac{1}{4}\frac{r^4}{a^4}\right)\right.$$
$$\left. + e^{-kpt}\sin pt\left[p\alpha\mu\left(\frac{1}{2} - \frac{r^2}{a^2}\right) + 2kp^2\alpha^2\mu^2\left(\frac{1}{6} - \frac{1}{2}\frac{r^2}{a^2} + \frac{1}{4}\frac{r^4}{a^4}\right) \&c.\right].$$

If we put this into the shape

$$Ae^{-kpt}\cos(pt + \beta),$$

A will be a quasi-amplitude of the disturbance at distance r from the axis. Its value works out

$$1 + \tfrac{1}{2}kp\alpha\mu\left(1 - \frac{2r^2}{a^2}\right) + \alpha^2\mu^2\left[k^2p^2\left(\frac{1}{6} - \frac{r^2}{2a^2} + \frac{r^4}{4a^4}\right) + \tfrac{1}{4}p^2\left(\frac{r^4}{a^4} - \frac{1}{6}\right)\right]$$

Now if $kp = 0$ while p is finite, so that there are undamped oscillations, then the last term in the last bracket is the first (besides the initial unit term) which does not vanish; and we see that its value falls off as r decreases—showing the ordinary *surface* concentration. If, on the other hand, the

damping is so great as to make the second term of the series more important than those which follow, then the value of A will increase as r decreases from a—showing *axis*-concentration. The coefficient of $kp\alpha\mu$ changes sign at $r = \frac{a}{\sqrt{2}}$. Assuming the preponderance of this term, this means that in the neighbourhood of this value we pass from a greater value on the inner parts to a less value on the outer, than would correspond to a uniform distribution throughout the wire.

From general reasoning it seems clear that if we think of a rapidly damped disturbance propagated into a wire from its boundary; and if the alternations are slow enough to allow the currents to penetrate to the core, we should expect to find an axial concentration during the latter stages of the phenomenon.

It may be of interest in this connexion to mention another case in which alternation and decay act in opposite ways as regards inductance. If oscillatory currents are being kept up in a primary coil, it is well known that the presence of a secondary decreases the effective inductance. But suppose that we have a steady current existing in a primary, that we cut off the applied E.M.F., and allow the current to die away. Then the nearness of a secondary coil will cause it to decay less rapidly, which corresponds to an *increase* of the apparent inductance. The case is worked out in Prof. J. J. Thomson's 'Elements of Electricity and Magnetism,' p. 391. It is there shown that the presence of a secondary of resistance S and inductance N changes the exponential which expresses the decay of current in the primary from

$$e^{-\frac{R}{L}t} \text{ to } e^{\frac{-R}{L\left(1+\frac{RN}{LS}\right)}t}.$$

Discussion.

Dr. Lehfeldt said that Prof. Lodge had pointed out, at the reading of the paper, that the solution the authors obtained changed character at the critical resistance. As this point had not been considered in the note, he supposed that the change in character made no difference to the results obtained.

XLVII. *On the Effect of a Solid Conducting Sphere in a Variable Magnetic Field on the Magnetic Induction at a Point outside.* *By* C. S. WHITEHEAD, *M.A.**

THIS paper is an investigation of the magnetic induction at a point outside a solid conducting sphere when magnetic disturbances are taking place in the dielectric outside.

From the equations obtained, it is shown that when the sphere becomes an infinite plate and the inducing system consists of an alternating current in a circular circuit whose plane is parallel to the surface of the plate, we have for a point just outside the plate and on the axis of the circuit,

$$R_d = \frac{6\pi\gamma\mu d}{k_1 f^3}\ ; \quad T_d = 0;$$

and for a point at a considerable distance from the axis and just outside the plate,

$$R_d = \frac{9\pi\gamma\mu d f^2}{k_1 \rho^5},$$

$$T_d = 3\left(2\pi\gamma d + \frac{\pi\gamma\mu}{k_1}\right)\frac{f^2}{\rho^4},$$

where R_d is the maximum value of the magnetic induction normal to the surface,

T_d the maximum value of the induction tangential to the surface,

f the radius of the inducing circuit,

d the distance of the inducing circuit from the surface of the plate,

ρ the distance of the point from the axis,

γ the maximum current in the inducing circuit,

μ the magnetic permeability of the plate,

σ its specific resistance,

$p = 2\pi \times$ frequency,

$k_1 = \left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}$.

From these equations, and with the values of p, μ, and σ

* Read April 21, 1899.

given in the paper, the value of R_d for a sea-water plate is 44 times as great as it would be for an iron plate, and more than 3000 times as great as it would be for a copper plate.

It is also seen that for the purpose of Induction Telegraphy the receiving coil should have its plane vertical and not horizontal to get the best effect; firstly because d must in practice be small compared with ρ, and hence R_d is small compared with T_d; and secondly because R_d varies inversely as ρ^5 while T_d varies inversely as ρ^4.

Let P, Q, R be the components of the electromotive intensity,

a, b, c the components of the magnetic induction,

σ the specific resistance of the sphere,

μ the magnetic permeability of the sphere.

Let P, Q, R all vary as $e^{\iota pt}$, where $p = 2\pi \times$ frequency and $\iota = (-1)^{\frac{1}{2}}$.

Neglecting the polarization-current, P, Q, R (in the sphere) satisfy

$$\nabla^2 \mathrm{P} = \frac{4\pi\mu}{\sigma} \frac{d\mathrm{P}}{dt},$$

$$\&c. = \quad \&c.$$

Let
$$k^2 = -\frac{4\pi\mu\iota p}{\sigma} = (1-\iota)^2 k_1^2,$$

where
$$k_1 = \left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}};$$

$$\therefore \nabla^2 \mathrm{P} + k^2 \mathrm{P} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with similar equations for Q and R. We also have

$$\frac{d\mathrm{P}}{dx} + \frac{d\mathrm{Q}}{dy} + \frac{d\mathrm{R}}{dz} = 0,$$

The most general solution of these equations is given by (*cf.* Recent Researches, art. 370), omitting the time factor,

$$\mathrm{P} = \Sigma \left\{ (n+1) f_{n-1}(kr) \frac{d\omega_n'}{dx} - nk^2 r^{2n+3} f_{n+1}(kr) \frac{d}{dx} \frac{\omega_n'}{r^{2n+1}} \right\}$$

$$+ \Sigma f_n(kr) \left(y \frac{d}{dz} - z \frac{d}{dy} \right) \omega_n. \quad . \quad . \quad . \quad . \quad (2)$$

Q and R satisfy similar equations. ω_n', ω_n are arbitrary

solid spherical harmonics of degree n, and $f_n(kr)$ satisfies

$$\frac{d^2f_n}{dr^2}+\frac{2(n+1)}{r}\frac{df_n}{dr}+k^2f_n=0. \quad . \quad . \quad . \quad (3)$$

In the sphere r can vanish, and we must take that solution of (3) which does not become infinite when r is zero,

$$\left.\begin{aligned} \therefore\ f_n(kr)&=\left(\frac{1}{kr}\frac{d}{d\,.\,kr}\right)^n\frac{\sin kr}{kr} \\ \text{also}\qquad \frac{1}{kr}\frac{d\,.\,f_n}{d\,.\,kr}&=f_{n+1} \\ k^2r^2f_{n+1}+(2n+1)f_n+f_{n-1}&=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Now the currents induced in the sphere are entirely due to magnetic forces outside the sphere, therefore (Recent Researches, art. 319) there can be no radial currents in the sphere, and therefore the radial electromotive intensity must vanish.

Hence the origin being at the centre of the sphere, we must have

$$\frac{x}{r}\mathrm{P}+\frac{y}{r}\mathrm{Q}+\frac{z}{r}\mathrm{R}=0\ ;$$

therefore from (2)

$$\frac{n(n+1)}{r}\{f_{n-1}(kr)+k^2r^2f_{n+1}(kr)\}\omega_n'=0,$$

$$\therefore\ -\frac{n(n+1)(2n+1)}{r}f_n(kr)\,.\,\omega_n'=0\ ;$$

therefore $\omega_n'=0$.

Hence, restoring the time-factor,

$$\left.\begin{aligned} \mathrm{P}&=\Sigma f_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\omega_n e^{ipt} \\ \mathrm{Q}&=\Sigma f_n(kr)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\omega_n e^{ipt} \\ \mathrm{R}&=\Sigma f_n(kr)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\omega_n e^{ipt} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

Now

$$\left.\begin{aligned} -\frac{da}{dt} &= \frac{d\mathrm{R}}{dy} - \frac{d\mathrm{Q}}{dz} \\ -\frac{db}{dt} &= \frac{d\mathrm{P}}{dz} - \frac{d\mathrm{R}}{dx} \\ -\frac{dc}{dt} &= \frac{d\mathrm{Q}}{dx} - \frac{d\mathrm{P}}{dy} \end{aligned}\right\}; \quad \ldots\ldots \quad (6)$$

therefore from (5)

$$\left.\begin{aligned} a &= -\frac{e^{\iota pt}}{\iota p(2n+1)}\left\{(n+1)f_{n-1}(kr)\frac{d\omega_n}{dx} - nk^2r^{2n+3}f_{n+1}(kr)\frac{d}{dx}\frac{\omega_n}{r^{2n+1}}\right\} \\ b &= -\frac{e^{\iota pt}}{\iota p(2n+1)}\left\{(n+1)f_{n-1}(kr)\frac{d\omega_n}{dy} - nk^2r^{2n+3}f_{n+1}(kr)\frac{d}{dy}\frac{\omega_n}{r^{2n+1}}\right\} \\ c &= -\frac{e^{\iota pt}}{\iota p(2n+1)}\left\{(n+1)f_{n-1}(kr)\frac{d\omega_n}{dz} - nk^2r^{2n+3}f_{n+1}(kr)\frac{d}{dz}\frac{\omega_n}{r^{2n+1}}\right\} \end{aligned}\right\}. \quad (7)$$

Equations (5) and (7) hold in the sphere.

In the dielectric, neglecting the displacement-current, a, b, c may be derived from a potential; we may therefore assume

$$\left.\begin{aligned} a &= \left\{\frac{d\Omega_n}{dx} + \mathbf{a}^{2n+1}\frac{d}{dx}\frac{\Omega_n'}{r^{2n+1}}\right\}e^{\iota pt} \\ b &= \left\{\frac{d\Omega_n}{dy} + \mathbf{a}^{2n+1}\frac{d}{dy}\frac{\Omega_n'}{r^{2n+1}}\right\}e^{\iota pt} \\ c &= \left\{\frac{d\Omega_n}{dz} + \mathbf{a}^{2n+1}\frac{d}{dz}\frac{\Omega_n'}{r^{2n+1}}\right\}e^{\iota pt} \end{aligned}\right\} \quad \ldots \quad (8)$$

where $\mathbf{a}$ = radius of the sphere, and Ω_n, Ω_n' are solid harmonics of degree n, to be determined from the nature of the inducing system.

Let R_s = component of the magnetic induction along the radius vector in the sphere.

R_d = the component in the dielectric.

T_s = component of the magnetic induction tangential to a curve at right angles to the radius vector in the sphere.

T_d = the component in the dielectric.

$$R_s = \frac{x}{r}a + \frac{y}{r}b + \frac{z}{r}c,$$

$$T_s = a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds};$$

and similarly for R_d and T_d.

Hence from (7) and (8) we find, after some reductions,

$$\left.\begin{aligned} R_s &= \frac{n(n+1)e^{\iota pt}}{\iota pr} f_n(kr)\omega_n \\ R_d &= \frac{e_{\iota pt}}{r}\left\{ n\Omega_n - (n+1)\left(\frac{a}{r}\right)^{2n+1}\Omega_n' \right\} \\ T_s &= -\frac{e^{\iota pt}}{\iota p}\left\{ f_{n-1}(kr) + nf_n(kr) \right\}\frac{d\omega_n}{ds} \\ T_d &= e^{\iota pt}\left\{ \frac{d\Omega_n}{ds} + \left(\frac{a}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds} \right\} \end{aligned}\right\} \quad . \quad . \quad (9)$$

The values of P, Q, R in the sphere are given by (5), their values in the dielectric may be found from (6) and (8). We obtain

$$P = \iota p e^{\iota pt}\left\{ \frac{1}{n+1}\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\Omega_n - \frac{1}{n}\left(y\frac{d}{dz} - z\frac{d}{dy}\right)\left(\frac{a}{r}\right)^{2n+1}\Omega_n' \right\}$$

$$Q = \iota p e^{\iota pt}\left\{ \frac{1}{n+1}\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\Omega_n - \frac{1}{n}\left(z\frac{d}{dx} - x\frac{d}{dz}\right)\left(\frac{a}{r}\right)^{2n+} \Omega_n' \right\}$$

$$R = \iota p e^{\iota pt}\left\{ \frac{1}{n+1}\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\Omega_n - \frac{1}{n}\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\left(\frac{a}{r}\right)^{2n+1}\Omega_n' \right\}$$

Let $\mathfrak{T}_s$ be the component of P, Q, R tangential to the same curve as before in the sphere.

$\mathfrak{T}_d$ the component in the dielectric.

Let $\frac{dx}{ds'}, \frac{dy}{ds'}, \frac{dz}{ds'}$ be the direction cosines of a curve perpendicular to both the radius vector and the curve ds.

$$\therefore \quad x\frac{dx}{ds'} + y\frac{dy}{ds'} + z\frac{dz}{ds'} = 0$$

$$\frac{dx}{ds}\cdot\frac{dx}{ds'} + \frac{dy}{ds}\cdot\frac{dy}{ds'} + \frac{dz}{ds}\cdot\frac{dz}{ds'} = 0.$$

Therefore

$$\frac{\frac{dx}{ds'}}{z\frac{dy}{ds} - y\frac{dz}{ds}} = \frac{\frac{dy}{ds'}}{x\frac{dz}{ds} - z\frac{dx}{ds}} = \frac{\frac{dz}{ds'}}{y\frac{dx}{ds} - x\frac{dy}{ds}} = \frac{1}{r}.$$

Hence

$$\left.\begin{aligned} \mathfrak{T}_s &= e^{\iota pt} r f_n(kr)\frac{d\omega_n}{ds'} \\ \mathfrak{T}_d &= \iota p e^{\iota pt} r\left\{\frac{1}{n+1}\frac{d\Omega_n}{ds'} - \frac{1}{n}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds'}\right\} \end{aligned}\right\}. \quad (10)$$

Let $\mathfrak{T}_s'$, $\mathfrak{T}_d'$ be the components of P, Q, R along ds' in the sphere and dielectric respectively.

We find as before

$$\left.\begin{aligned} \mathfrak{T}_s &= e^{\iota pt} r f_n(kr)\frac{d\omega_n}{ds} \\ \mathfrak{T}_d' &= \iota p e^{\iota pt} r\left\{\frac{1}{n+1}\frac{d\Omega_n}{ds} - \frac{1}{n}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds}\right\} \end{aligned}\right\}. \quad (10^*)$$

At the surface of the sphere, the normal magnetic induction is continuous, and also the tangential magnetic force.

Therefore

$$\mathrm{R}_s = \mathrm{R}_d \text{ when } r = \mathrm{a},$$

$$\frac{1}{\mu}\mathrm{T}_s = \mathrm{T}_d \quad ,, \quad r = \mathrm{a}.$$

Therefore

$$\frac{n(n+1)}{\iota p} f_n(k\mathrm{a})\omega_n = n\Omega_n - (n+1)\Omega_n'$$

$$-\frac{1}{\mu\iota p}\{f_{n-1}(k\mathrm{a}) + n\ {}_n\ k\mathrm{a})\}\omega_n = \Omega_n + \Omega_n'.$$

We see that the first of these equations also arises from the continuity of the tangential electromotive intensity.

Eliminating ω_n,

$$(n+1)\Omega_n' = -\frac{(\mu n+\mu+n)f_n(k\mathrm{a})+f_{n-1}(k\mathrm{a})}{(\mu-1)nf_n(k\mathrm{a})-f_{n-1}(k\mathrm{a})}\,n\Omega_n . \quad (11)$$

Now

$$f_n(kr)=\left(\frac{1}{kr}\frac{d}{d\,.\,kr}\right)^n\frac{\sin kr}{kr}$$

$$=(-)^n\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(kr)^{-(n+\frac{1}{2})}J_{n+\frac{1}{2}}(kr),$$

where $J_{n+\frac{1}{2}}(kr)$ is Bessel's function.

Hence when kr is large,

$$f_n(kr)=(kr)^{-(n+1)}\left\{P_n\sin\left(kr+\frac{n\pi}{2}\right)+Q_n\cos\left(kr+\frac{n\pi}{2}\right)\right\};$$

where $$P_n=1-\frac{(n-1)n(n+1)(n+2)}{2\,!\,(2kr)^2}+\ldots.$$

$$Q_n=\frac{n(n+1)}{2kr}-\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{3\,!\,(2kr)^3}+\ldots$$

the real part of k is to be positive, therefore $k=(1-\iota)k_1$.

Let kr be so large that we may take

$$P_n=1, \qquad Q_n=\frac{n(n+1)}{2kr};$$

therefore

$$f_n(kr)=(kr)^{-(n+1)}\frac{e^{k_1r}e^{\iota(k_1r+\frac{n\pi}{2})}}{2\iota}\left\{1-\frac{n(n+1)}{2\iota kr}\right\},$$

$$\therefore \quad \frac{f_{n-1}(kr)}{f_n(kr)}=e^{-\frac{\iota\pi}{2}}\,kr\,\frac{1-\frac{(n-1)n}{2\iota kr}}{1-\frac{n(n+1)}{2\iota kr}}$$

$$=-\iota kr\left(1+\frac{n}{\iota kr}\right).$$

$$\therefore \quad (n+1)\Omega_n'=\frac{\iota k\mathrm{a}-\mu(n+1)}{\iota k\mathrm{a}+\mu n}\,n\Omega_n$$

$$=\left\{1-\frac{\mu(2n+1)}{\iota k\mathrm{a}}\right\}n\Omega_n. \quad . \quad (12)$$

$$\therefore \quad (n+1)\Omega_n' = \left\{1 - \frac{\mu(2n+1)}{2k_1\mathrm{a}} + \iota\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\} n\Omega_n.$$

$$\therefore \quad n\Omega_n - (n+1)\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\Omega_n'$$

$$= n\Omega_n\left[1 - \left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}\right.$$

$$\left. - \iota\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right]$$

$$= n\Omega_n\xi_n(\cos\zeta_n - \iota\sin\zeta_n),$$

where

$$\tan\zeta_n = \frac{\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{\mu(2n+1)}{2k_1\mathrm{a}}}{1 - \left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}}$$

$$\xi_n^2 = \left[1 - \left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}\right]^2 + \left\{\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}^2.$$

$$\therefore \quad \xi_n = 1 - \left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}.$$

Now R_d = real part of $\frac{e^{\iota pt}}{r}\left\{n\Omega_n - (n+1)\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\Omega_n'\right\}$

$$= \ldots\ldots\ \frac{e^{\iota pt}}{r}n\Omega_n\xi_n e^{-\iota\zeta_n}$$

$$= \frac{\xi_n n\Omega_n}{r}\cos(pt - \zeta_n).$$

$$\therefore \quad \mathrm{R}_d = \Sigma\frac{n}{r}\left\{1 - \left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left(1 - \frac{\mu(2n+1)}{2k_1\mathrm{a}}\right)\right\}\cos(pt - \zeta_n)\Omega_n. \quad . \quad (13)$$

Again,

$$\frac{d\Omega_n}{ds} + \left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds}$$

$$= \frac{d\Omega_n}{ds}\left[1 + \frac{n}{n+1}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1 - \frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}\right.$$

$$\left. + \iota\cdot\frac{\mu n(2n+1)}{(n+1)2k_1\mathrm{a}}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\right],$$

$$= \eta_n\left(\cos\phi_n + \iota\sin\phi_n\right)\frac{d\Omega_n}{ds};$$

where

$$\tan\phi_n = \frac{\dfrac{\mu n(2n+1)}{(n+1)2k_1\mathrm{a}}\left(\dfrac{\mathrm{a}}{r}\right)^{2n+1}}{1+\dfrac{n}{n+1}\left(\dfrac{\mathrm{a}}{r}\right)^{2n+1}\left\{1-\dfrac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}}$$

$$\eta_n^2 = \left[1+\frac{n}{n+1}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1-\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\}\right]^2 + \left[\frac{\mu n(2n+1)}{(n+1)2k_1\mathrm{a}}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\right]^2;$$

$$\therefore \quad \eta_n = 1+\frac{n}{n+1}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left\{1-\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right\};$$

$$\mathrm{T}_d = \text{real part of } e^{\iota pt}\left\{\frac{d\Omega_n}{ds}+\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\frac{d\Omega_n'}{ds}\right\}$$

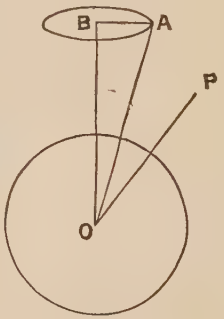
$$= \ldots\ldots\ldots e^{\iota pt}\eta_n e^{\iota\phi_n}\frac{d\Omega_n}{ds};$$

$$\therefore \quad \mathrm{T}_d = \Sigma\left\{1+\frac{n}{n+1}\left(\frac{\mathrm{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathrm{a}}\right)\right\}\cos(pt+\phi_n)\frac{d\Omega_n}{ds} \quad . \quad (14)$$

In these formulæ (13) and (14), for R_d and T_d it is assumed that Ω_n is wholly real; this is the case in the application we are going to make.

Let the magnetic field outside the sphere be due to an alternating current in a circular circuit whose axis passes through the centre of the sphere.

Fig. 1.



Let O be the centre of the sphere,
B the centre of the circuit,
A any point on the circuit,
P any point.

Let $\mathrm{BA}=f$; $\mathrm{OA}=c$;
$\mathrm{OP}=r$;
$\angle\mathrm{BOA}=a$; $\angle\mathrm{BOP}=\theta$.

Let Ω be the solid angle subtended by the circuit at P.
$\gamma e^{\iota pt}$ the current in the circuit,
V the potential due to the circuit at P,
U the magnetic force along OP.

Therefore $$V=\gamma e^{\iota pt}\Omega,$$

$$U=-\frac{dV}{dr}=-\gamma e^{\iota pt}\frac{d\Omega}{dr}.$$

Now $$\Omega=2\pi\left\{1-\cos\alpha+\sum_1^\infty \frac{\sin^2\alpha}{n}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta)\right\}.$$

$r \angle c$.

P_n is the zonal harmonic of the n^{th} order, and

$$P_n'=\frac{dP_n}{d\mu},\quad \mu=\cos\theta.$$

$$\therefore\quad U=-2\pi\gamma e^{\iota pt}\sum_1^\infty \frac{\sin^2\alpha}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta).$$

In the dielectric magnetic induction is the same thing as magnetic force.

Therefore, from the second equation of (9)

$$\Omega_n=-2\pi\gamma\frac{\sin^2\alpha}{n}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta).$$

Therefore

$$\frac{d\Omega_n}{rd\theta}=2\pi\gamma\frac{\sin^2\alpha}{n}\frac{\sin\theta}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n'(\theta).$$

Therefore, when $k\mathbf{a}$ is large,

$$R_d=-2\pi\gamma\sum_1^\infty\left[1-\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right)\right]$$
$$\times\frac{\sin^2\alpha}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n(\theta)\cos(pt-\zeta_n);\quad\ldots\quad(15)$$

$$T_d=2\pi\gamma\sum_1^\infty\left[1+\frac{n}{n+1}\left(\frac{\mathbf{a}}{r}\right)^{2n+1}\left(1-\frac{\mu(2n+1)}{2k_1\mathbf{a}}\right)\right]$$
$$\times\frac{\sin^2\alpha}{n}\frac{\sin\theta}{r}\left(\frac{r}{c}\right)^n P_n'(\alpha)P_n'(\theta)\cos(pt+\phi_n),\quad\ldots\quad(16)$$

if $ds=rd\theta$.

If r is but little greater than $\mathbf{a}$,

$$\tan \zeta_n = 1, \quad \text{therefore } \zeta_n = \frac{\pi}{4} \text{ approximately;}$$

$$\tan \phi_n = 0, \quad \text{therefore } \phi_n = 0 \text{ approximately.}$$

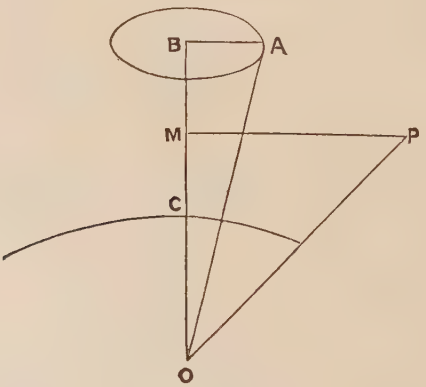
Therefore

$$R_d = -2\pi\gamma \cos\left(pt - \frac{\pi}{4}\right) \sum_1^\infty \left[\left\{ 1 - \left(\frac{\mathbf{a}}{r}\right)^{2n+1} \left(1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right) \right\} \frac{\sin^2\alpha}{r} \left(\frac{r}{c}\right)^n P_n'(\alpha) P_n(\theta) \right];$$

$$T_d = 2\pi\gamma \cos pt \sum_1^\infty \left[\left\{ 1 + \frac{n}{n+1} \left(\frac{\mathbf{a}}{r}\right)^{2n+1} \left(1 - \frac{\mu(2n+1)}{2k_1\mathbf{a}}\right) \right\} \times \frac{\sin^2\alpha}{n} \frac{\sin\theta}{r} \left(\frac{r}{c}\right)^n P_n'(\alpha) P_n'(\theta) \right].$$

Let the radius of the sphere become infinite, the sphere becomes a plate, and we must change from spherical harmonics to Bessel's functions.

Fig. 2.



Let $BC = d$,

$BM = z$, $\quad PM = \rho$,

$\sin\theta = \frac{\rho}{r}$, $\quad \sin\alpha = \frac{I}{c}$

Let $n = \lambda r = \lambda c$,

where λ is to remain finite when n, r, and c become infinite.

$$\mathbf{a}=c-d, \qquad r=c-z \text{ ultimately};$$

$$\left(\frac{\mathbf{a}}{r}\right)^{2n+1}=e^{-2\lambda(d-z)}; \qquad \left(\frac{r}{c}\right)^{n}=e^{-\lambda z};$$

$$\mathrm{P}_n(\theta)=\mathrm{J}_0(\lambda\rho); \qquad \mathrm{P}_n'(\theta)=\frac{\lambda r^2}{\rho}\mathrm{J}_1(\lambda\rho);$$

$$\mathrm{P}_n'(a)=\frac{\lambda c^2}{f}\mathrm{J}_1(\lambda f).$$

$$\frac{\mu(2n+1)}{2k_1\mathbf{a}}=\frac{\mu\lambda}{k_1}, \quad rd\lambda=1.$$

Hence

$$\mathrm{R}_d=-2\pi\gamma f\int_0^\infty\lambda\left\{e^{-\lambda z}-\left(1-\frac{\mu\lambda}{k_1}\right)e^{-\lambda(2d-z)}\right\}\mathrm{J}_0(\lambda\rho)\mathrm{J}_1(\lambda f)\cos(pt-\zeta_n)d\lambda;$$

$$\mathrm{T}_d=2\pi\gamma f\int_0^\infty\lambda\left\{e^{-\lambda z}+\left(1-\frac{\mu\lambda}{k_1}\right)e^{-\lambda(2d-z)}\right\}\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda f)\cos(pt+\phi_n)d\lambda;$$

so that for a point just outside the plate,

$$\mathrm{R}_d=-\frac{2\pi\gamma f\mu}{k_1}\cos\left(pt-\frac{\pi}{4}\right)\int_0^\infty\lambda^2e^{-\lambda d}\mathrm{J}_0(\lambda\rho)\mathrm{J}_1(\lambda f)d\lambda, \quad \ldots (17)$$

$$\mathrm{T}_d=2\pi\gamma f\cos pt\int_0^\infty\left(2-\frac{\mu\lambda}{k_1}\right)\lambda e^{-\lambda d}\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda f)d\lambda. \quad \ldots (18)$$

Now (*cf.* Gray and Mathews, 'Treatise on Bessel's Functions,' p. 27)

$$\mathrm{J}_0\sqrt{b^2+2bc\cos\alpha+c^2}=\mathrm{J}_0(b)\mathrm{J}_0(c)-2\mathrm{J}_1(b)\mathrm{J}_1(c)\cos\alpha+2\mathrm{J}_2(b)\mathrm{J}_2(c)\cos2\alpha-\ldots\ldots$$

Write $\lambda\rho$ for b and λf for c, multiply both sides by $e^{-\lambda d}$ and integrate with respect to λ from $\lambda=0$ to $\lambda=\infty$.

$$\therefore \quad \int_0^\infty e^{-\lambda d}\mathrm{J}_0(\lambda\rho)\mathrm{J}_0(\lambda f)d\lambda-2\int_0^\infty e^{-\lambda d}\mathrm{J}_1(\lambda\rho)\mathrm{J}_1(\lambda f)d\lambda\cos\alpha+\ldots\ldots$$

$$=\int_0^\infty e^{-\lambda d}\mathrm{J}_0\{\lambda\sqrt{\rho^2+f^2+2\rho f\cos\alpha}\}d\lambda.$$

$$=\frac{1}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos\alpha}}$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}}$$

$$+ \frac{2}{\pi}\sum_1^{\infty} \cos n\alpha \int_0^{\pi} \frac{\cos nv\, dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}}$$

by Fourier's theorem;

$$\therefore \int_0^{\infty} e^{-\lambda d} J_0(\lambda\rho) J_0(\lambda f)\, d\lambda = \frac{1}{\pi}\int_0^{\pi} \frac{dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}} \quad . \quad (19)$$

$$\int_0^{\infty} e^{-\lambda d} J_n(\lambda\rho) J_n(\lambda f)\, d\lambda = \frac{(-)^n}{\pi}\int_0^{\pi} \frac{\cos nv\, .\, dv}{\sqrt{d^2+\rho^2+f^2+2\rho f\cos v}} \quad (20)$$

Differentiate the first of these equations with respect to f. Since

$$\frac{dJ_0(\lambda f)}{df} = -\lambda J_1(\lambda f),$$

$$\int_0^{\infty} \lambda e^{-\lambda d} J_0(\lambda\rho) J_1(\lambda f)\, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{(f+\rho\cos v)dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{3}{2}}} \quad . \quad (21)$$

Differentiate again with respect to d.

$$\therefore \int_0^{\infty} \lambda^2 e^{-\lambda d} J_0(\lambda\rho) J_1(\lambda f)\, d\lambda$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{3d(f+\rho\cos v)dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}} \quad . \quad (22)$$

Differentiate (20) with respect to d.

$$\int_0^{\infty} \lambda e^{-\lambda d} J_n(\lambda\rho) J_n(\lambda f)\, d\lambda$$

$$= \frac{(-)^n}{\pi}\int_0^{\pi} \frac{d\,.\cos nv\,.\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{3}{2}}} \quad . \quad (23)$$

Differentiating again with respect to d.

$$\int_0^{\infty} \lambda^2 e^{-\lambda d} J_n(\lambda\rho) J_n(\lambda f)\, d\lambda$$

$$= \frac{(-)^n}{\pi}\int_0^{\pi} \frac{-(\rho^2+f^2+2\rho f\cos v-2d^2)\cos nv\, dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}}\ ; \quad (24)$$

therefore, for a point in the dielectric just outside the plate,

$$\left.\begin{aligned} R_d &= -\frac{2\gamma\mu f}{k_1}\cos\left(pt-\frac{\pi}{4}\right)\int_0^\pi \frac{3d(f+\rho\cos v)\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}} \\ T_d &= -4\gamma f\cos pt\int_0^\pi \frac{d\,.\cos v dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{3}{2}}} \\ &\quad -\frac{2\gamma\mu f}{k_1}\cos pt\int_0^\pi \frac{(\rho^2+f^2+2\rho f\cos v-2d^2)\cos v\,dv}{(d^2+\rho^2+f^2+2\rho f\cos v)^{\frac{5}{2}}} \end{aligned}\right\}. \quad (25)$$

R_d is the magnetic induction perpendicular to the surface of the plate.

T_d the magnetic induction tangential to the surface along ρ.

Let the point be on the axis of the circuit, so that $\rho=0$. Therefore

$$\left.\begin{aligned} R_d &= -\frac{6\pi\gamma\mu}{k_1}\frac{df^2}{(d^2+f^2)^{\frac{5}{2}}}\cos\left(pt-\frac{\pi}{4}\right) \\ T_d &= 0. \end{aligned}\right\} \quad . \quad . \quad (26)$$

Let ρ be greater than f and large compared with d, so that d^2 may be neglected.

We have

$$(1+2h\cos v+h^2)^{-\frac{s}{2}} = (1+he^{iv})^{-\frac{s}{2}}(1+he^{-iv})^{-\frac{s}{2}}$$

$$= (1+A_1he^{iv}+A_2h^2e^{2iv}+\ldots\ldots+A_mh^me^{miv}+\ldots.)$$

$$\times(1+A_1h_1e^{-iv}+A_2h^2e^{-2iv}+\ldots.+A_mh^me^{-miv}+\ldots.),$$

where

$$A_m=(-)^m\frac{s(s+2)\ldots.(2m+s-2)}{2^m\,.\,m!}$$

$$A_0=1.$$

The coefficient of h^m

$$=2A_m\cos mv+2A_1A_{m-1}\cos(m-2)v+2A_2A_{m-2}\cos(m-4)v$$

$$+\ldots.$$

the last term $=A^2_{\frac{m}{2}}$, if m be even

$$=2A_{\frac{m-1}{2}}A_{\frac{m+1}{2}}\cos v, \text{ if } m \text{ be odd.}$$

We hence find

$$\left.\begin{aligned} R_d &= \frac{9\pi\gamma\mu f^2 d}{k_1\rho^5}\cos\left(pt-\frac{\pi}{4}\right)\left\{1+\frac{1}{2}\binom{5}{2}^2\left(\frac{f}{\rho}\right)^2\right. \\ &\qquad \left. +\frac{1}{3}\left(\frac{5.7}{2^2.2!}\right)^2\left(\frac{f}{\rho}\right)^4+\ldots.\right\} \\ T_d &= \left\{4\pi\gamma d+\frac{2\pi\gamma\mu}{k_1}\right\}\frac{f^2}{\rho^4}\cos pt\left\{\frac{3}{2}+\binom{3}{2}^2\frac{5}{4}\left(\frac{f}{\rho}\right)^2\right. \\ &\qquad \left. +\left(\frac{3.5}{2^2.2!}\right)^2\frac{7}{6}\left(\frac{f}{\rho}\right)^4+\ldots.\right\} \end{aligned}\right\}. \quad (27)$$

Hence approximately when d is small compared with f and ρ large compared with f, we have, considering only maximum values, for a point on the axis

$$R_d=\frac{6\pi\gamma\mu}{k_1}\frac{d}{f^3}\,;\quad T_d=0,\quad .\quad .\quad .\quad .\quad (28)$$

and for a point a considerable distance from the axis

$$\left.\begin{aligned} R_d &= \frac{9\pi\gamma\mu}{k_1}\frac{f^2 d}{\rho^5} \\ T_d &= 3\left(2\pi\gamma d+\frac{\pi\gamma\mu}{k_1}\right)\frac{f^2}{\rho^4} \end{aligned}\right\}. \quad .\quad .\quad .\quad .\quad .\quad (29)$$

It appears from these equations that when the primary circuit is near the surface of the plate the normal magnetic induction at a point just outside the plate is small, but that the tangential magnetic induction at a point some distance from the axis is not necessarily small ; hence for the purpose of " Induction Telegraphy " the receiving-circuit should be vertical rather than horizontal.

Let $p=1885$, which makes the frequency 300 ; then for a plate in which $\mu=1$, $\sigma=2\,.\,10^{19}$, which are about the values for sea-water according to Mr. S. Evershed's experiments,

$$k_1=\cdot00077\,;$$

for $\mu=1$, $\sigma=1600$, which are about the values for copper,

$$k_1=2\cdot72;$$

for $\mu=1000$, $\sigma=10^4$, which are about the values for iron,

$$k_1=34\cdot4\,;$$

for a sphere the size of the earth $a = 637 \times 10^6$ cms. ; so that in all these cases $k_1 a$ is large ; hence for a sea-water plate the magnetic induction normal to the surface is 44 times as great as it would be were the plate of iron, and more than 3000 times as great as it would be were the plate of copper.

Let C_s' be the current in the sphere along the curve s'

$$C_s' = \frac{\Sigma'_s}{\sigma};$$

when ka is large,

$$\omega_n = \frac{\mu(2n+1)p}{(n+1)ka} \frac{\Omega_n}{f(ka)};$$

therefore from 10 * ,

$$C_s' = \frac{\mu p(2n+1)}{(n+1)ka\sigma} \frac{r f_n(kr)}{f_n(ka)} \frac{d\Omega_n}{ds} e^{ipt}$$

$$= \frac{\sqrt{2}k_1(2n+1)}{4\pi(n+1)} \left(\frac{a}{r}\right)^n e^{-k_1(a-r)} \left\{ 1 - \frac{n(n+1)(a-r)}{4k_1 ar} \right\}$$

$$\times \cos\{pt - k_1(a-r) + \chi\} \frac{d\Omega_n}{ds}, \quad \ldots \quad (30)$$

where $$\tan\chi = \frac{\frac{1}{2k_1}}{\frac{1}{2k_1} - \frac{n(n+1)(a-r)}{4k_1^2 ar}}$$

Hence $\tan\chi = 1$ approximately,

therefore $\chi = \frac{\pi}{4}$.

Hence when the inducing system is the circular circuit before mentioned, $ds = r d\theta$, and we find

$$C' = \frac{\sqrt{2}\gamma k_1}{2} e^{-k_1(a-r)} \cos\left\{ pt - k_1(a-r) + \frac{\pi}{4} \right\}$$

$$\times \sum_1^\infty \frac{\sin^2\alpha \sin\theta}{r} \left(\frac{a}{c}\right)^n \frac{2n+1}{n(n+1)} \left\{ 1 - \frac{n(n+1)}{4k_1} \frac{a-r}{ar} \right\} P_n'(\alpha) P_n'(\theta). \quad (31)$$

Discussion.

Mr. Blakesley observed that, as a rule, experiment preceded theory. He congratulated Mr. Whitehead upon having settled from theoretical considerations that the vertical position of the receiving coil is best.

Prof. Everett said that the very elaborate method of analysis adopted in the paper appeared to be very clearly stated. He would like to know whether the inducing coil ought to be vertical as well as the receiving coil.

Mr. Appleyard thought that the experiment had left no doubt as to the best position of both coils. The early investigations of Mr. Willoughby Smith and the later work of Mr. H. R. Kempe and Mr. Preece had proved that for the best effect both coils should be vertical. But large vertical coils were difficult to fix and expensive to maintain. It was this reason probably that led Prof. Lodge to try what could be done with coils placed horizontally.

Mr. Whitehead, in reply, said that his final formulæ only applied to a horizontal inducing coil. He had not worked out the case of what would happen if the inducing coil itself was vertical. In the Flat Holm experiments both circuits were straight wires with their ends to earth, so that they really amounted to vertical coils.

XLVIII. *Note on the Vapour-Pressure of Solutions of Volatile Substances.* *By* R. A. Lehfeldt, *D.Sc.**

The change of vapour-pressure of a solvent due to the solution in it of a small quantity of volatile material has been calculated on the basis of Raoult's rule for the corresponding case of a non-volatile dissolved body, first by Planck, and later, more correctly, by Nernst; the formulæ obtained, and one experiment in support of them, are given in Ostwald's *Lehrbuch* (2. ii. 588). Further examples may be obtained from the measurements of vapour-pressures I have carried out on certain mixtures of organic liquids†; those measurements

* Read May 12, 1899.

† Phil. Mag. [5] vol. xlvi. p. 46.

were made primarily with the object of following out the behaviour of liquids in solutions not dilute; but in the extreme cases the results obtained will serve as a test of the formulæ applying to dilute solutions.

Adopting the notation used in the memoir referred to, we have for a mixture of the two liquids A and B :—

π_A, π_B, vapour-pressure of A and B respectively,
p, ,, mixture
ζ, ratio of number of mols. of A to mols. of B in liquid,
η, ,, ,, ,, vapour.

Then the relation as expressed by Nernst becomes

$$\frac{\pi_B - p}{\pi_B} = \zeta - \frac{p}{\pi_B}\eta$$

for the case in which ζ is small, *i. e.* in which a small quantity of A is dissolved in B; π_B is therefore the vapour-pressure of the solvent. This equation may more conveniently be transformed into

$$\frac{p}{\pi_B} = \frac{1-\zeta}{1-\eta}. \quad \ldots\ldots\ldots \quad \text{(i.)}$$

When ζ, and consequently η, is very small, this becomes

$$\frac{p}{\pi_B} = 1 - \zeta + \eta,$$

the form in which Planck first gave the result, but which is less exact than the other when moderate concentrations are considered.

The expression in the above form (i.) is so simple that it is worth while to put it into words, thus :—

"*When a volatile substance is dissolved in a liquid the vapour-pressure of the liquid is altered in the ratio of the molecular fractional amount of solvent* (*Molenbruch*) *in the liquid to that in the vapour.*"

When the dissolved substance is non-volatile, the latter fraction becomes unity, and we recover the well-known formula $p/\pi_B = 1 - \zeta$.

It is perhaps worth remarking that the expression con-

sidered is a particular case of the thermodynamical relation applicable to any liquid mixtures, obtained by Duhem, Margules, and myself. The latter may be put * :—

$$(\eta-\zeta)\frac{\partial \log \frac{\eta}{1-\eta}}{\partial \log \frac{\zeta}{1-\zeta}} = \frac{\zeta(1-\eta)}{p}\frac{\partial p}{\partial \zeta},$$

whence

$$\frac{\eta-\zeta}{\eta(1-\eta)}\frac{\partial \eta}{\partial \zeta} = \frac{1}{p}\frac{\partial p}{\partial \zeta}.$$

If we integrate this for a small range from $\zeta=0$ and therefore $p=\pi_B$ we get

$$\frac{(\eta-\zeta)\eta}{\eta(1-\eta)} = \frac{p-\pi_B}{\pi_B},$$

or

$$\frac{p}{\pi_B}\eta-\zeta = \frac{p-\pi_B}{\pi_B},$$

the formula in question.

In case of a dilute solution of B in A it is easily seen that equation (i.) becomes

$$\frac{p}{\pi_A} = \frac{\zeta}{\eta} \quad \ldots\ldots\ldots \text{(ii.)}$$

The experiments made were on four series of mixtures, viz. :—alcohol with benzene and toluene, and carbon tetrachloride with benzene and toluene; so that, taking the end members of each series, we get eight cases of "dilute solution." The second group of liquids, which were chosen as normal in character, give the following results :—

	η.	π (solvent).	p.	ζ (calc.).	ζ (obs.).
CCl_4 in toluene	0·181	93·0	105·4	0·072	0·073
Toluene in CCl_4	0·872	310·2	242·0	0·680	0·708
(A = carbon tetrachloride; B = toluene.)					
CCl_4 in benzene	0·054	270·9	273·6	0·045	0·043
Benzene in CCl_4.....	0·835	310·2	306·5	0·825	0·830
(A = carbon tetrachloride; B = benzene.)					

* *L. c.* p .60.

Unfortunately no sufficiently dilute solution of toluene in CCl_4 was measured; that quoted contained (as may be seen from the value of ζ) 29·2 per cent. of the dissolved body, and it appears that the range of applicability of the formula is here exceeded: otherwise the agreement is good.

The mixtures containing alcohol, on the other hand, show maxima of vapour-pressures, and on this account the departure from the formulæ for dilute solutions is much more marked, and it is not possible to apply equations (i.) and (ii.) when the amount of dissolved body exceeds a very few per cent. The numbers are:—

	η.	π (solvent).	p.	ζ (calc.).	ζ (obs.).
Alcohol in benzene	0·281	270·9	350·4	0·070	0·088
Benzene in alcohol	0·668	219·5	315·0	0·958	0·886
(A = alcohol; B = benzene.)					
Alcohol in toluene	0·591	93·0	199·5	0·123	0·138
Toluene in alcohol	0·902	219·5	233·5	0 959	0·946
(A = alcohol; B = toluene.)					

Temperature, throughout = 50° C.

The quantity of dissolved substance in the liquid is less than that calculated, in each of the four alcohol mixtures. It is easy to show that for them the formula (i.) or (ii.) is less accurate than in the case of normal liquids, for a mixture in nearly equal parts may be looked upon either as a solution of A in B or of B in A. If it be calculated accordingly the results will of course differ, since the solution is not by any means dilute; but whilst in the case of the normal liquids the disagreement of the two results is small, in the alcohol mixtures it is egregious.

East London Technical College,
March 1899.

XLIX. *On the Thermal Properties of Normal Pentane.*—Part II.

By J. Rose-Innes, *M.A., B.Sc., and* Prof. Sydney Young, *D.Sc., F.R.S.**

[Plate I.]

In the first part of our paper on this subject, read before the Physical Society last December†, we published a large array of figures giving the relations between the volume, temperature, and pressure of normal pentane; it was suggested that a reasonably good agreement between calculation and experiment might be secured by using the formula

$$p=\frac{\mathrm{RT}}{v}\left\{1+\frac{e}{v+k-gv^{-2}}\right\}-\frac{l}{v(v+k)}$$

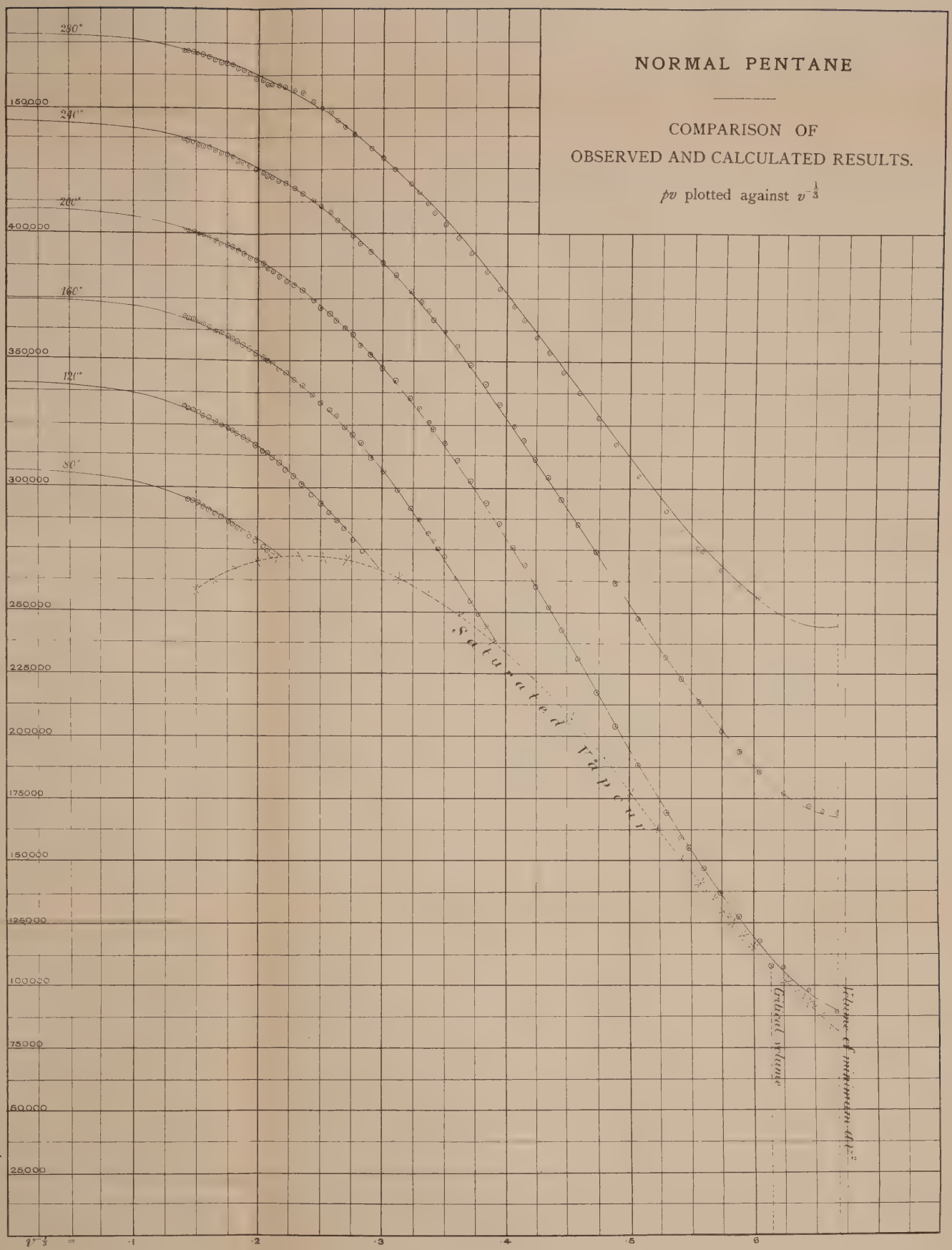
(*ante*, p. 336). This formula was first found by one of us for isopentane, but it seemed likely that it would do equally well for normal pentane; and arguments were adduced showing that in such case the values of R and of $l \div e$ might be taken to be the same for the two isomers. The question then arose whether l and e for normal pentane could be separately taken as equal to the values already found for isopentane, and the question was provisionally answered in the negative (*ante*, p. 337). Two methods were employed in dealing with this matter, but neither of them could be regarded as conclusive.

The first plan was to plot the differences of $\frac{1}{av^2}$ against $v^{-\frac{1}{3}}$; these differences grew smaller as the volume was made larger, but it did not seem likely that they vanished altogether when the volume was made infinite. The "wobbling" at large volumes, however, made it impossible to speak with any certainty one way or the other.

The second plan adopted was to take the three constants R, e, and l the same as for isopentane, and then to calculate suitable values for the two remaining constants k and g from the experimental data near the critical point. The isothermals

* Read May 26, 1899.

† Proc. Phys. Soc. present vol. p. 322; Phil. Mag. [5] vol. xlvii. p. 353.



Proc. Phys. Soc. Vol. 16 Pl. I.

were then calculated from the formula, and it was found that deviations occurred between calculation and experiment amounting to nearly 2 per cent. Accordingly the hypothesis on which the calculations were founded was considered to be most probably incorrect.

This second method of investigating the question has several advantages. We have to deal with the experimental data directly instead of indirectly as in the first method, and there is not any noticeable "wobbling" at large volumes. On the other hand, it is difficult to know how much of the deviation between calculation and experiment may be fairly attributed to experimental error, how much to the intrinsic imperfection of the formula, and how much to the special hypothesis employed. Further investigations were therefore undertaken, and the result of these forms the subject of the present communication.

After several ineffectual attempts, it was found possible to secure good concordance between calculation and experiment for volumes above 3·4 by means of the following hypothesis:— The values of R, of $l \div e$, and of g were taken as being the same respectively as those previously found for isopentane. The two constants l and k were given suitable values derived from the experimental data near the critical point. The values actually found were $l = 5{,}678{,}100$, $k = 3{\cdot}57$.

In order to test this hypothesis as thoroughly as possible a diagram was made in which pv was plotted against $v^{-\frac{1}{3}}$; the calculated isothermals were drawn as continuous lines while the experimental results were put in as dots; this diagram is reproduced on Plate I. An examination of it shows that the agreement between calculation and experiment is satisfactory; indeed the errors do not exceed 1 per cent. They are consequently less than those which occur in the tables published by one of us in conjunction with Prof. Ramsay as proving the truth of the linear law (Phil. Mag. xxiii. pp. 438–447). The improvement effected by means of the present hypothesis is so marked, that it is impossible to attribute it wholly or even chiefly to a compensation of errors: we may therefore regard our previous tentative hypothesis, that the value of l is the same for the two pentanes, as distinctly disproved. But we cannot assert, in the present state of the evidence, that our

present hypothesis is actually proved ; we can merely note that it introduces errors so small as to be comparable with the intrinsic imperfection of the characteristic equation.

Finally we may conclude that the difference of pressure between two isomeric substances at the same temperature and volume involves the same power of the density as the first deviation from Boyle's law ; *i. e.*, the second power.

Discussion.

Prof. Callendar expressed his interest in the wide applicability of the author's formula, and asked if any theoretical significance could be assigned to the various constants which appeared.

Mr. Rose-Innes said the R of their formula was the R of the perfect gas equation, and that the l and e corresponded respectively to the β and α of the ordinary van der Waals expression. So far as he knew the k and g had no physical interpretation.

L. *The Minor Variations of the Clark Cell.*

By A. P. Trotter.*

[Abstract.]

A number of Clark cells was adopted three years ago, together with a resistance-box, as an electrical standard for the Colony of the Cape of Good Hope. The sum provided for the equipment of the Government Electrical Laboratory was a small one, and this, coupled with the risk of damage to the instruments during the journey, led to the adoption of the potentiometer (Crompton type), with various auxiliary resistances, in preference to balances or other instruments containing mechanism.

The cells consist of seven Muirhead portable Clark cells ; the Board of Trade pattern with a considerable quantity of loose mercury not being considered to be so suitable. It was intended to compare the cells together periodically, and

* Read March 24, 1899.

to assume that if they kept in fair agreement they kept also fairly constant. It is intended also from time to time to supplement them by new cells. In addition to these Muirhead cells there are two Clark cells by Wolff with Reichsanstalt certificates, and two Carhart cells by Nalders.

Comparisons were made about once a month from July 1896 to February 1897, and then for about a year comparisons were made weekly. During the week ending April 3, 1897, comparisons were made twice a day. The object of this communication is to record the result of these comparisons, and to discuss the utility of the Clark cell as a laboratory standard.

The degree of precision aimed at is intermediate between that of high-class scientific measurements which are not likely to be required of the Government Electrical Laboratory of this Colony, and the commercial standard of 1 *in* 400 *or* $\frac{1}{4}$ *per cent.* The fact that the value of the Clark cell is not known absolutely with greater accuracy than 1 in 1000, has no more to do with the precision of these comparisons than the discrepancy of 0·018 mm. between the metre and the ten-millionth part of the quadrant has to do with the precision of ordinary measurements of length.

The interesting researches on the temperature-lag of Clark-cells*, and the considerable number of modifications of inverted †, dry ‡, chloride §, and unsaturated ‖ cells, which aim at reducing the temperature-coefficient or rendering it negligible for industrial purposes, have but little bearing on the use of the Clark cell for a laboratory standard; for the temperature of the cells can be easily kept sufficiently uniform to render the lag inappreciable. Whether the temperature coefficient be large or small is of no importance whatever, so long as it is definite.

The temperature of the Laboratory, when temporarily situated

* Ayrton and Cooper, Proc. Roy. Soc. vol. lix.; Electrician, xxxviii. p. 303; Spiers, Twyman, and Waters, Proc. Phys. Soc. xvi. p. 38.

† Callendar and Barnes, Rep. British Assoc. 1897; Electrician, xxxix. p. 638.

‡ Kahle, *Instrumentenkunde*, 1893, p. 298.

§ Hibbert, Electrician, xxxvii. p. 320.

‖ Carhart, Electrician, xxiv. pp. 271, 643.

in the South African College, varied from about 28° to 13° C. during 1896. The cells are placed in a wooden box within a larger box, a space of three inches between being filled with loosely packed waste telegraph-slip. It had been intended to fill this space with aerated paraffin-wax or some other non-conductor; but no further precaution seemed necessary in view of the ultimate destination of the cells. At the South African College, a recording thermometer on the table showed daily variations of about 2° to 2½° C., and when placed under the table on the box, about 0°·75 C. The gradient at 10 A.M., when the comparison was generally made, had been about 0°·03 C. per hour for the previous 16 hours. Inside the box the changes were considerably less, and being considered negligible, were not investigated.

The Laboratory is now situated in the basement of the General Post Office; the box is placed below the floor in a small well. A thermometer, graduated in tenths of a degree, is lowered through a hole into the double box. Having regard to the precision of absolute measurement aimed at, and the various minor corrections which were disregarded, it was not attempted to read it closer than to tenths of a degree. Although the temperature was recorded in all cases as it was read, the slide of the potentiometer was sometimes set to the nearest round number on the main slide (scale of volts), if not more than about one-tenth of a degree from the ascertained temperature. This enabled a check reading, which was taken between each comparison, to be effected more quickly. The temperature-scale of the potentiometer is not divided to tenths of a degree, these were estimated.

A recording thermometer placed on the double box, in the well, shows that the temperature rarely varies more than ⅓ of 1° C. in 24 hours. The changes within the box are, of course, much less.

Since the mean of such Clark cells as agree well together is assumed to be the standard, it was necessary, in making the comparisons, to select an arbitrary standard and to compare each cell in turn with it. Cell No. 1 was thus used. In consequence of the prelimary balancing (after a rough balance had been obtained with the "hack" cells), and the

check test of No. 1 for steadiness of potentiometer current between each measurement of a cell, No. 1 was used from twice to three times more than the others. It does not, however, appear, from comparison between this cell and the mean of the whole set, to have been affected appreciably. Cell No. 1 was assumed to have the value given on the potentiometer-scale according to the engraved scale of temperature; viz. 1·4396 at 10° C., 1·4342 at 15° C., 1·4286 at 20° C., and so on. The scale of the potentiometer was then assumed to be a scale of volts, and the other cells were measured on it.

As the difference between the cells rarely exceeds 0·001 volt, only a quarter of an inch of the slide-wire, which corresponds to this difference, was used during any one comparison. The slide-wire has been carefully examined. An additional slider was made, and various lengths of the wire were intercepted between them. The greatest error (for purposes of comparison) would have been 1·47993 instead of 1·48, or 4·7 parts in 100,000; but there was no appreciable error at the part of the wire used. In making the comparisons the readings were taken generally to the fourth decimal figure, or one-tenth of a millivolt; the $\frac{1}{20}$ and even the $\frac{1}{40}$ of a millivolt was occasionally estimated. Mr. W. L. Logeman, a student at the South-African College, rendered much assistance in these comparisons as well as in other work in the Laboratory. Independent comparisons of the cells by Mr. Logeman and by myself very seldom varied in the fourth decimal figure, and often agreed even to the estimated halving ($\frac{1}{20}$ millivolt) of a division.

It was noticed that some of the cells, particularly the large Wolff cells, gave, for a given displacement of the slider, greater galvanometer-deflexions than others. This was obviously due to differences of internal resistance, and this led to the regular measurement of the internal resistances whenever the cells were compared. The mode adopted was to displace the slider 5 divisions, so as to cause a want of balance of 0·0005 volt, and to observe the deflexion. The deflexion was the same whether the slider was so displaced as to allow the cell to discharge, or whether it received a small charge shunted from the potentiometer current. The deflexion was

inversely as the sum of the resistances of the cell and the galvanometer. A scale of resistances empirically graduated was attached to the galvanometer-scale. The resistance of the Wolff cells was too low to be measured with any precision in this way.

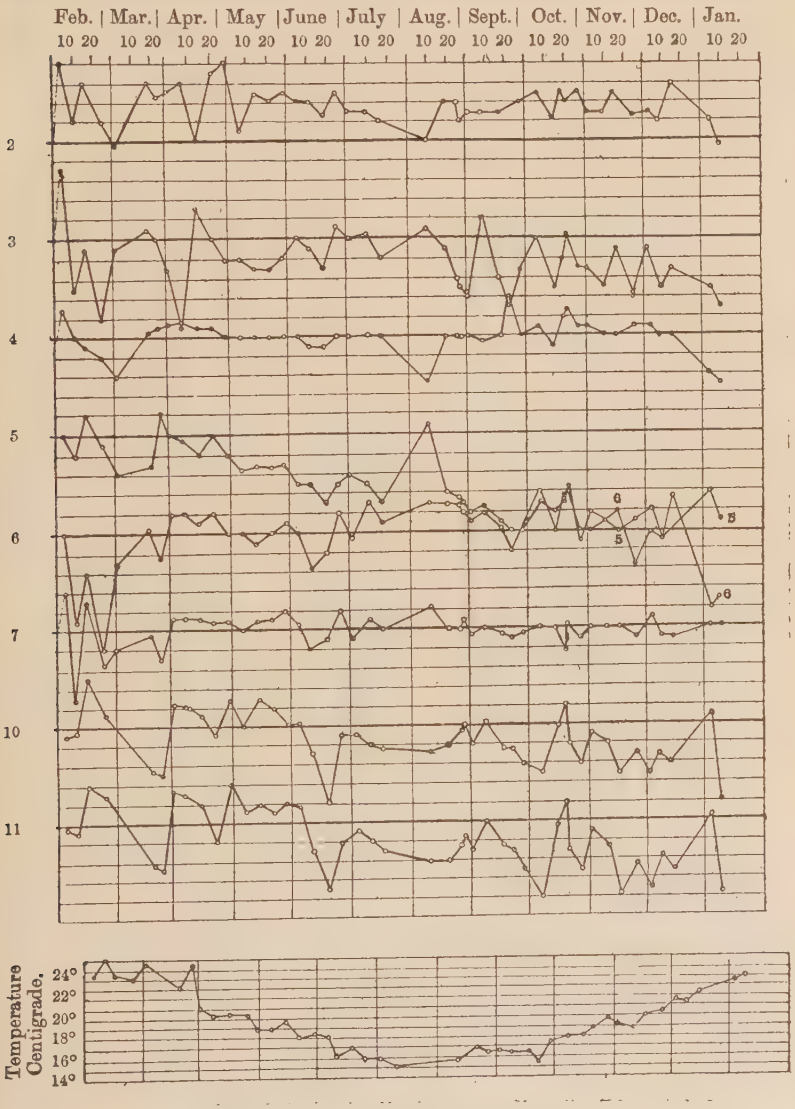
The results of the comparisons in 1897–98 are plotted in figs. 1 to 4. Fig. 1 shows the differences from cell No. 1. The heavy horizontal lines are zeros, and are numbered 2, 3, 4, &c., corresponding to the different cells. (Nos. 8 and 9 omitted, relate to the Carhart cells, which are not fairly comparable with the others.) The thinner horizontal lines represent differences of 0·0002 volt. The first observation is in each case joined to the corresponding zero line by a chain-dotted line, except in 5, 6, and 11, where little or no difference was found. Below this diagram is given a curve of the temperature of the cells. The most striking feature of this diagram is the close resemblance between the differences of the two large Wolff cells, Nos. 10 and 11; but a study of the lines representing the behaviour of the other cells leads at once to the question, What was No. 1 doing all this time?

In fig. 3 the differences from the mean of the nine cells are plotted. Here No. 1 appears on the same footing as the others. Nos. 10 and 11, while retaining their resemblance, are somewhat altered in detail. The first idea which occurs is that these variations are chiefly due to errors of observation and to minor corrigenda. But they are of quite another order. The probable error of setting and reading the slide of the potentiometer is about $\frac{1}{20}$ millivolt, represented by one quarter of the distance between two horizontal lines in the diagrams. Thermoelectric and plug-resistance comparison-errors, which must be distinguished from errors of measurement, are eliminated by the use of a null method.

The variations being so capricious, the cells were compared twice a day for a week, with the result shown in fig. 2. Even at such short intervals there is a lack of continuity in most of the curves; but the reduction in the variations between any two successive measurements is noteworthy.

The remarkable agreement of the Wolff cells together strongly suggests that these should be the standards. On the other hand, the curves 10 and 11, fig. 3, show that they

Fig. 1.—Differences from No. 1. One Division = 0·0002 volt = $\frac{1}{5}$ millivolt.



differ considerably from the mean of the other seven cells, for the mean on which fig. 3 is based is a mean weighted with these two cells themselves. They often differed by only a tenth of a millivolt, and seldom by more than a fifth of a

Fig. 2.—Differences from No. 1.
One Division = 0·0002 volt.

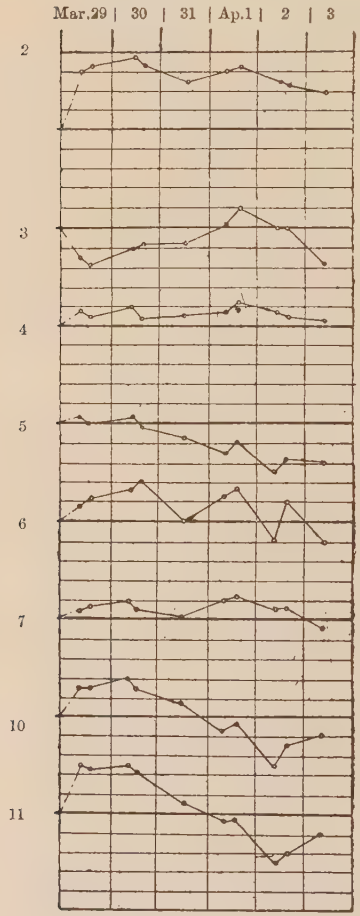
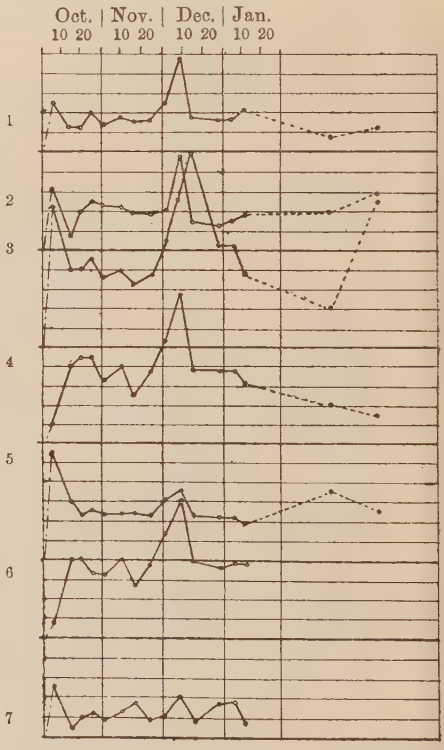


Fig. 4.—Internal Resistance.
One Division = 1000 ohms.

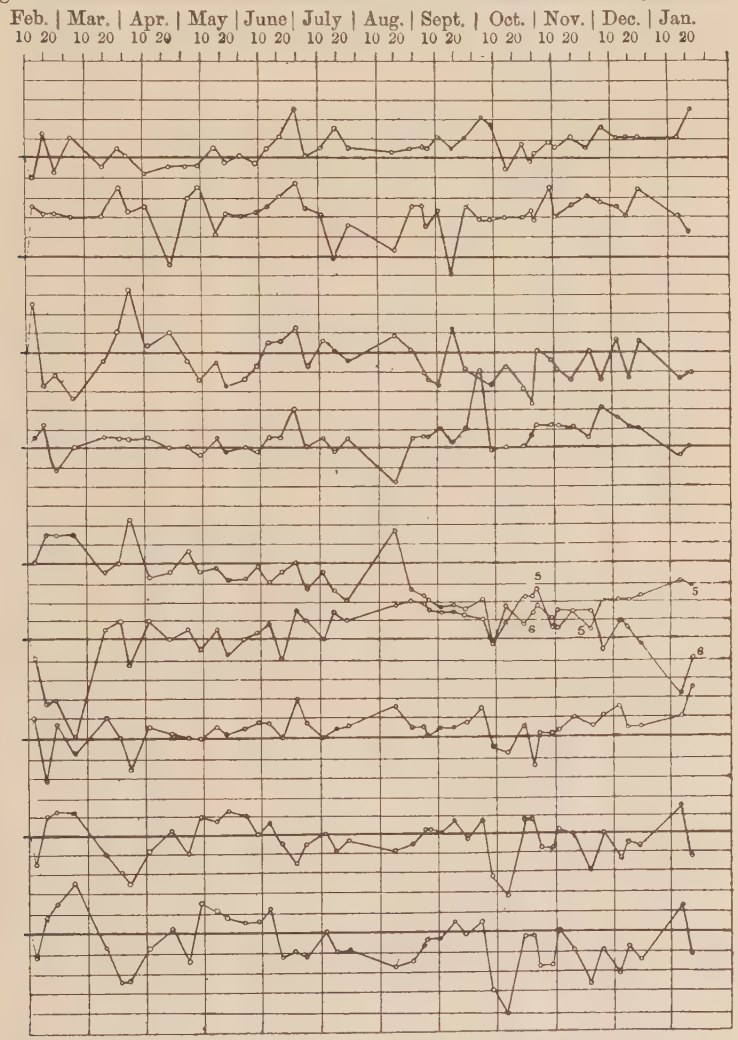


millivolt from each other, but on several occasions by more than half a millivolt from the mean of the whole.

A study of the variations of the other cells from the mean shows several runs of coincidence: for example, Nos. 1, 2, 4,

and 7 in June, Nos. 1, 4, and 6 in May. On the other hand, the contradictory variations are quite as well marked: for example, 3 and 5 with 6 and 7 in March, or 2 and 4 with

Fig. 3.—Differences from the Mean. One Div. = 0·0002 volt = $\frac{1}{5}$ millivolt.



3 and 5 during the interval which unfortunately occurred between July 16 and August 10th. The marked drop in Nos. 10 and 11 at the end of September is found also in

No. 7, and to a less extent in 4, 5, and 6 ; but for the majority of the variations no coincidences can be discerned. There appears to be sufficient continuity in the curves to suggest that the internal state of the cells is by no means so quiescent as might be imagined. Perhaps the only general result is the falling off of No. 5, but no long series of comparisons was needed to show that.

The ease with which the internal resistance of the cells could be measured was only discovered in October, and so far as the observations went, no relation is apparent between the voltage and the resistance. The result is plotted in fig. 4. It is probable that the measurement on December 9 was faulty, but it must be remarked that there was an upward tendency among all the cells, especially 4 and 6. Hot weather was beginning, and there had been a slight fall of temperature, but no change sufficient to account for the discontinuity.

So far from being able to offer an explanation of these variations, I am unable to make any generalization about them or even to give a systematic description of them.

Two opportunities have occurred for comparison with newer Clark cells. One of these was 0·4 millivolt, and the other 0·6 millivolt lower than the mean of the standards. It is not improbable that this mean is now somewhat too high, as the cells are more than three years old; and there appears to be a tendency for Clark cells to rise a little in the course of time. *It should be observed, however, that the value of the mean of these cells in absolute measure is not the subject of the present communication.*

The capriciousness of the variations which have been described have a fortunate result, in that they appear to be on the whole as likely to be above as below the mean. The approximate probable difference of any cell from the mean is 0·23 millivolt; and the approximate probable deviation of the observed mean of the nine cells from the true mean is 0·077 millivolt or 5·4 parts in 100,000.

It appears, therefore, that the Clark cell used under conditions which may be easily attained in any permanent laboratory is of great value, at all events as a secondary standard, and that its value is worthy to be known to 1 part in 10,000 instead of 1 in 1000.

Discussion.

Mr. E. H. Griffiths said that the paper appeared to have value only in so far that it showed that Clark cells at Cape Town behaved in a manner that agreed with common knowledge and general experience everywhere else. Their variations depended upon shifts of temperature, and the consequent changes in the degree of saturation of the liquid. His own experiments during seven years, upon 42 Clark cells, had shown that if the temperature was kept constant to within 0·01° C., the steadiness and uniformity of all the E.M.F.'s was most remarkable. They started with discrepancies, which diminished with lapse of time until they became negligible. It was of little use to put a thermometer anywhere but within the cells; very slight changes of temperature caused serious changes in the degree of saturation of the liquid. The existence of the capricious lag of E.M.F. behind temperature precluded the possibility of formulating a satisfactory temperature correction for the ordinary form of Clark cells. In the case of Callendar cells there was no lag; the E.M.F. varied with temperature, by a definite amount, which could be corrected by a coefficient.

Mr. W. R. Cooper said the method of comparison used by the author was unsuitable, because to arrive at the differences of E.M.F. necessitated the measurement of the E.M.F. of each cell. The variations only amounted to a few ten-thousandths of a volt. The length of potentiometer-wire corresponding to a thousandth of a volt was only a quarter of an inch; under such conditions it would be difficult to ensure accuracy. A method of opposition would have been preferable. Mr. Cooper had found that Board of Trade cells only vary about one ten-thousandth of a volt between themselves from day to day under ordinary conditions. Cells of the H form vary about one-fifth of that amount.

Mr. A. P. Trotter (communicated). It appears that my desire to investigate with care the precision yielded by the simple means and apparatus which I have chosen, has been confused with a desire to arrive at a high degree of precision; a result which is sometimes important but often useless.

The degree of precision which was aimed at has been over-

looked in the discussion. I consider that the use of thermostats, water-baths, stirrers, ice, separate thermometers in each cell, and other refinements with which I am familiar, would, under the present circumstances, be quite out of place. I have the greatest regard for high precision in its proper place, as for example in the researches of Mr. Griffiths, but an accuracy of one part in a thousand is far in excess of any practical requirements of my official work. The whole equipment of the Government Electrical Laboratory, including storage-battery and testing sets for outdoor inspections, had to be provided for the sum of £300. I supplemented this with instruments and new purchases of my own. I consider that the potentiometer which I use is amply accurate for the requirements of this laboratory. It is probable that the correction of temperature errors of the potentiometer, thermoelectric and contact errors are quite as important as the use of a thermometer in each cell, and none of these are warranted by the circumstances. To adopt some of the refinements which have been suggested would, in this case, be as unintelligent as the use of seven-figure logarithms for work in which the data are only known to 2 or 3 per cent.

It is true that no precautions were taken to keep the temperature of the selected cell constant. So to use a comparison cell would be to make a standard of it. Why, then, have the other eight? My method is to use all the nine cells, treating none as being more worthy of being standards than others. The method is unusual, but my circumstances are unusual. I made no *measurements* of potential difference of the cells, for I have no other standard than the mean of the cells. I prefer to call this a comparison rather than a measurement. Having arrived at the mean, I am prepared to measure other differences of potential by it.

Bearing in mind that my object was a comparison, and not a measurement of a potential difference in terms of the length of a wire, details of the area of the slider-contact are of small importance, but as they have been asked for, I give them.

By slider-contact is probably meant the length of wire in contact with the slider, for the area has nothing to do with the matter. The brass knife-edge has become worn with use, and there is now a groove about $\frac{1}{3}$ mm. long axially,

and of a considerably flatter curvature than that of the wire, which is about 0·2 mm. in diameter (it is not worth while to disturb it to measure it exactly). The contact is made by the ingenious "let off" key of the Crompton potentiometer, with a definite pressure of between 1 and 2 grammes, irrespective of the pressure on the key. In making a measurement of the electromotive force of a cellin terms of the length of a wire, a knife-edge would be advantageous ; but the limits of experimental error of the whole apparatus may be represented by a $\frac{1}{8}$length of the wire; and any width of knife-edge considerably less than that length is a useless refinement. The present paper, however, does not discuss measurements of electromotive force, but of differences between cells. For that purpose, the length of wire in contact with the slider may be very much larger than in the former case without affecting the accuracy of the comparison. Similarly, since the temperature-coefficient is a linear function (within my limits of temperature), a mistake of a whole degree in reading the thermometer would not appreciably affect the comparisons.

I carefully considered the method of comparing the cells in pairs back to back, and decided, for reasons which need not be discussed here, that for the purpose of obtaining the mean of the whole set with the precision desired, the potentiometer method was better.

Intentionally, simple conditions were used ; such conditions as may easily be realized in the test-room of engineering works, or in other Colonial laboratories ; and the precision *of comparison* actually obtained, under these conditions, say 6 in 100,000, appears to be very satisfactory, and is much higher than I expected.

The allusion to the paper by Ayrton and Cooper on the "Variations in the Electromotive Force of the Clark Cell with Temperature," was accompanied by the remark that this and certain other papers have but little bearing on the use of the Clark cell for a laboratory standard. In the research recorded in that paper, the cells were taken through cycles of about 10° in $2\frac{1}{2}$ hours, and only one small and slow cycle of 2° at the rate of 1° in 50 min. is described. These

are of interest in the case of portable cells carried about for inspections &c., or for cells left on a laboratory table with no temperature precautions. The temperature range at the South African College was 1° during 24 hours, and the temperature gradient was 0°·02 per hour for about 18 hours previous to the comparison. Such a gradient can be easily obtained in a more temperate climate than that of Cape Town.

No generalization is made in Ayrton and Cooper's paper on the connection between the magnitude of the lag and the temperature gradients. Only two gradients are given, viz., 4° per hour and 1·2° per hour. The difference between the rising and the falling volts appears to be about 15 and 4 ten-thousandths of a volt respectively, or, say, 2·8 millivolt per 1° per hour. From only two cases it cannot be determined whether the lag is simply proportional to the gradient; but assuming that that is the case, a gradient of 0·°02 per hour would give a lag of 0·056 millivolt, or an error of 0·028 millivolt, assuming the true value to lie midway between the rising and the falling curve. I went into this calculation at an early stage in the work, but did not allude to it in the paper otherwise than by the observation that the changes of temperature "being considered negligible were not investigated." Mr. Griffiths's remark that the variations of Clark cells depend on shifts of temperature does not appear to have much bearing on the variations which form the main subject of the paper, since the latter are, if the intrapolation of Ayrton and Cooper's results may be taken, about 15 times greater than the variations due to temperature. These variations are of the same order as the errors of observation shown in Ayrton and Cooper's diagrams. To account for these variations by differences of temperature between the cells, is to assume differences of from one-half to one degree between different cells. Under the conditions of the double box, lagging, and small temperature gradient observed, it is inconceivable that such differences could exist.

LI. *On the Distribution of Magnetic Induction in a Long Iron Bar.* *By* C. G. Lamb, *M.A.*, *B.Sc.** (Communicated by Prof. Ewing.)

The following investigation was undertaken to determine the distribution of induction in a long cylindrical iron rod when it was subjected to various magnetizing forces. It was felt that the great variation of induction which must necessarily occur in a cylindrical bar would vitiate to some extent the ordinary assumptions made when employing rods for magnetic measurements.

In order to get a sufficiently clear notion of the induction distribution, and how it is produced, it is desirable to determine the following data:—

(P) A curve of magnetization of the bar as determined by means of a search-coil at its centre.

(Q) A series of curves at various fixed magnetizing forces showing the distribution of induction in each case.

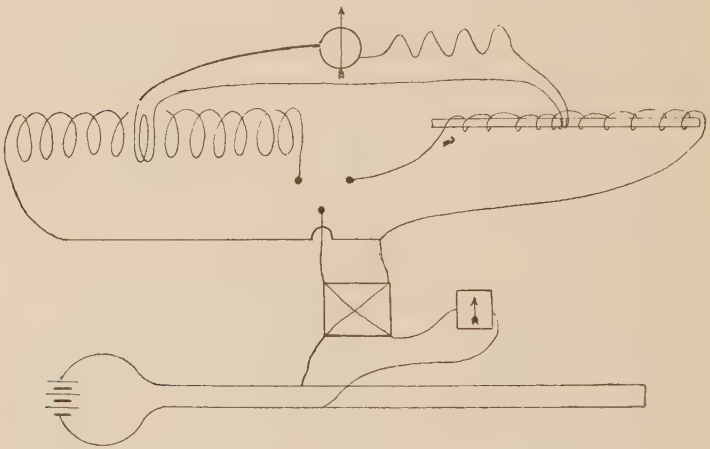
(R) A magnetization-curve of the bar when made into a ring, so that the induction is the same at every cross-section.

The bar used in the present case was a circular bar of Lowmoor iron; it was 0·485 cm. in mean diameter, and 123·4 cm. (48 inches) long, and before experiment was very carefully annealed. The magnetizing-coil was wound on a brass tube, somewhat longer than the specimen and of larger diameter; the specimen was fixed centrally inside this by ebonite rings near the centre and at one end. The search-coil was made as small as possible, and was wound on an ebonite bobbin attached to the end of another brass tube of such a size as just to slide over the rod, while the search-coil could just slide inside the tube covering the magnetizing-coil: on the inner tube marks were made showing when the search-coil was at the centre of the bar, and at other definite positions up to the end. For the determination of curve D it was kept at the centre. The larger magnetizing-currents were read on Weston standard ammeters calibrated by comparison with a

* Read June 9, 1899.

Crompton potentiometer; the small currents and those on the steep parts of the magnetization-curves were read directly by means of the potentiometer. The currents were adjusted by ordinary resistances for the larger values, or by a simple circular potential-slide as shown developed in fig. 1 for the smaller ones.

Fig. 1.



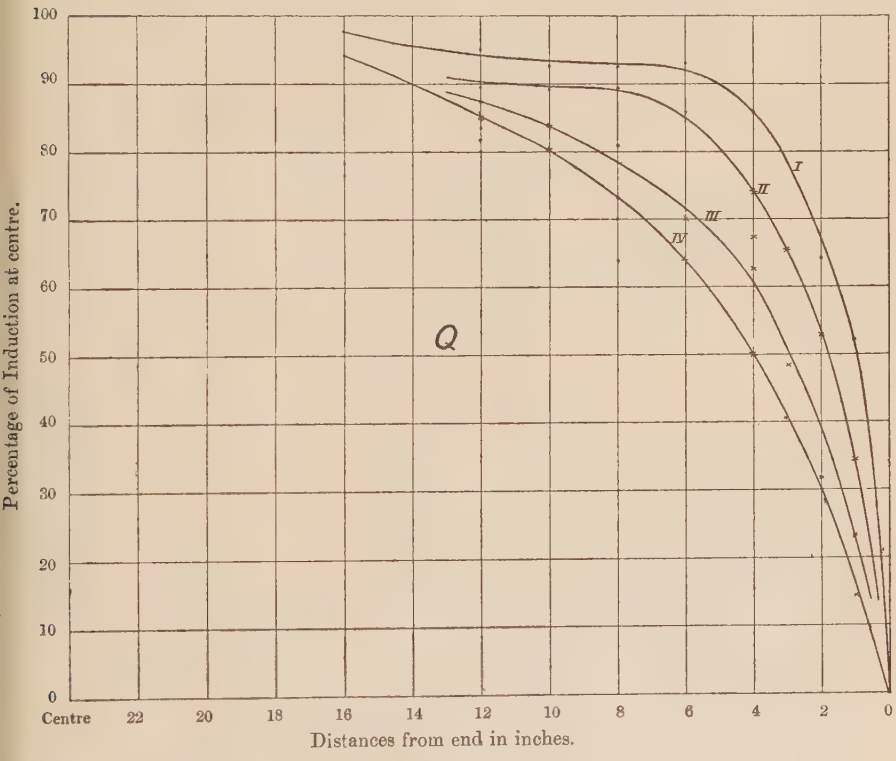
The ballistic galvanometer used was a Crompton "Midget" D'Arsonval, and was tested for proportionality by means of a standard field of known mutual-induction coefficient; it was found to be practically quite accurate. The standard field was also used throughout the experiment to calibrate the ballistic galvanometer, the connexions being shown in fig. 1. Fig. 2 shows the arrangement of the search-coil and specimen rod.

Fig. 2.

The results on reduction were carefully plotted to a large scale, and fig. 5 is a reduced copy of this curve; it connects the applied magnetizing force and the induction at the centre of the bar; we will refer to these as H_a and B_c.

The next step was to determine the curves of induction in the bar; these are called set Q (figs. 3 and 4), and nine were taken (numbered I. to IX.) at values of the magnetizing-current giving values of H_a shown in Table II. The points selected at which to measure the induction were: as near the end as possible, 1, 2, 3, 4, 6, 8, 10, 12, 14, 16, 18, 20 inches from that end, and at the centre (24 inches from end). A preliminary

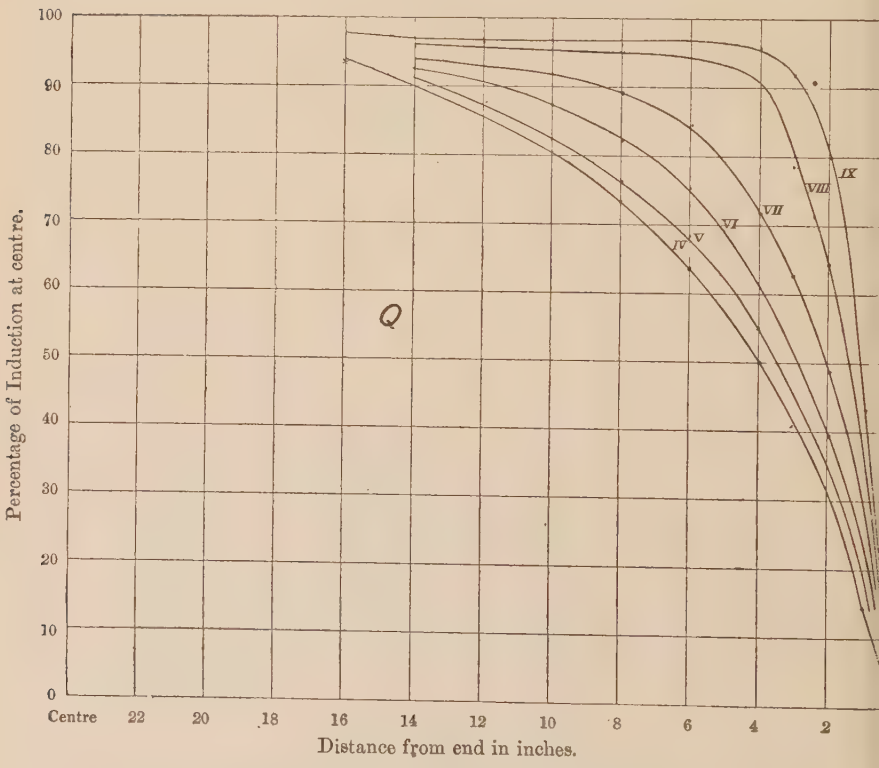
Fig. 3.



experiment showed that either half of the bar gave closely agreeing results at one definite value of H_a. Several observations were taken at each point, and from the mean of these the induction at each of the specified points was calculated as a percentage of that which occurred at the centre: these

values are given in Table I. Curves were carefully plotted to a large scale; those marked Q are given to show the mode of variation along the bar, the points near the centre are not shown as it will be noted from Table I. that some of the curves cross there owing to slight local differences in the iron; these points are omitted to avoid confusion. The curves subsequently referred to as "Curves Q" must be

Fig. 4.



taken as being drawn carefully through all the points, not those given here.

The bar was then very carefully welded and re-annealed, wound with a secondary coil and then a magnetizing-coil, and a reversal-curve again taken as for the bar. B, H curves were

Table I.

No. of Curve Q.	Inches from end.												
	Centre.	20.	18.	16.	14.	12.	10.	8.	6.	4.	3.	2.	1.
I. ...	100	.....	99	96	94	95	93	93	93	86	78	64	52
II. ...	100	.....	99	95	91	89·5	89·5	89·5	86	74	60	53	34
III. ...	100	.....	100	97	92	87·5	83·5	78	70	67·5	48	37	23
IV. ...	100	.....	99	94·1	90·5	85	80·2	73·3	63·7	50	40·5	31·8	14·3
V. ..	100	99·4	96·2	93	90·3	87·2	82·6	76·4	68	53·8	45	34	21·4
VI. ...	100	98·8	96·2	93·6	91·8	90·1	87·2	82·4	75	61	51·5	39·5	24·5
VII. ...	100	98·5	96	94	93·5	93	92	89	84·5	71·5	62	48·5	31
VIII. ...	100	99·0	99	96·5	96·2	96	95·5	95	94·5	92	78·5	64	41·5
IX. ...	100	99·8	98·5	97·5	97·5	96·5	96·5	96	97	96	92	80	43

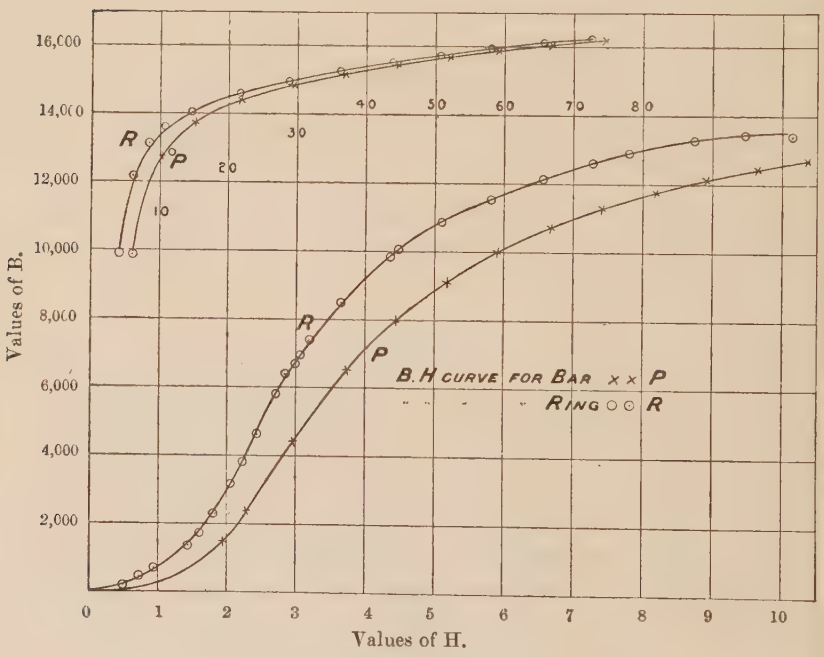
Table II.

Curve Q.	H_a.	H_m.	B_c.	B_m.	B_h.	$\frac{H_m}{H_a}$.	$\frac{B_m}{B_c}$.	$\frac{B_h}{B_m}$.	Equiv. length as fraction.	D in inches.
I. ...	0·74	.....	162	146	... ..	.....	.....	.....	·90	2·4
II. ...	1·49	1·18	572	477	468	·79	·84	·98	·835	4·0
III. ...	2·23	1·81	2,350	1,880	1,790	·81	·80	·96	·80	4·8
IV. ...	3·35	2·30	5,500	4,070	4,080	·69	·74	1·02	·74	6·3
V. ...	4·47	2·84	8,000	6,160	6,200	·64	·77	1·00	·77	5·4
VI. ...	6·70	3·84	10,700	8,680	8,800	·57	·81	1·01	·81	4·6
VII. ...	11·6	6·04	13,500	11,900	11,740	·52	·88	·99	·88	2·9
VIII. ...	20·0	11·50	14,300	12,900	13,600	·58	·90	1·04	·90	2·3
IX. ...	35·0	22·10	15,200	14,100	14,600	·63	·92	1·04	·92	1·9

drawn to suitable scales for high and low inductions so as to permit accurate reading of the quantities; the curve connecting B and μ was deduced and likewise plotted. Reduced copies of these curves are shown at R (figs. 5 and 6). References after must be taken to apply to the large-scale curves as before.

From curves Q we can easily deduce the mean induction by taking the area and base of the curves; this is given in Table II. under the heading B_m. The equivalent length of

Fig. 5.

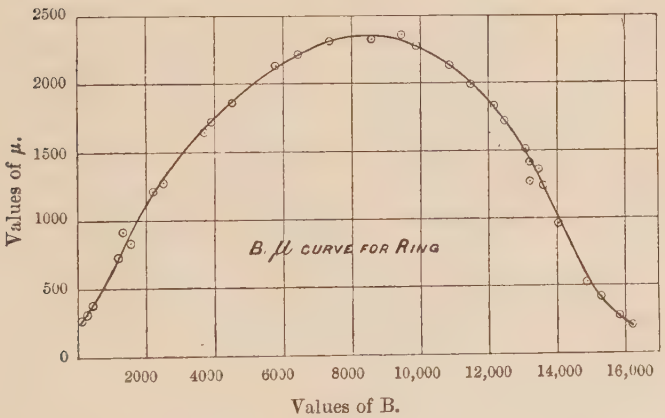


the bar will be the same proportion of its actual length that B_m is of B_e; this is also shown in the table. Half the difference between the true and the equivalent lengths will give the distance of the resultant pole from the end; this is tabulated under the heading D.

From Table I. and the curve P the actual inductions can be derived for the various points of the bar at any definite

value of H_a; if we can assume the annealing to have been good enough to permit us to take the bar as being the same under the two conditions (which is borne out by the close approximation of the curves towards the maximum induction), it is possible to deduce the distribution of H in the bar by cross reading from curve R the value of the H for each value of B at each point. Curves showing the distribution of H are given in set S (fig. 7). From these the value of the mean H was found as above described for B, and is tabulated under H_m in Table II. The induction this would have produced in the bar

Fig. 6.



was likewise read off curve R, and is given in Table II. under B_h; it is the induction that would have been produced by the mean value of H had the induction been uniform all along the bar.

The values of μ at successive points were likewise read off from curve, and these are shown in curves T (fig. 8).

Table II. contains also the ratios of some of the more important quantities.

On comparing P and R (fig. 5) it will be noticed that the differences in the abscissæ (giving the "demagnetizing force" at the centre) are far from proportional to B, until we get to regions where the induction through the bar is so high as to be

Fig. 7.

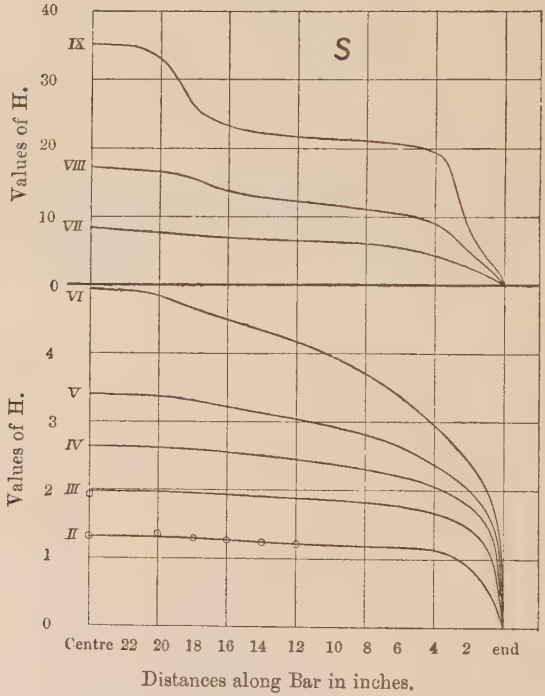
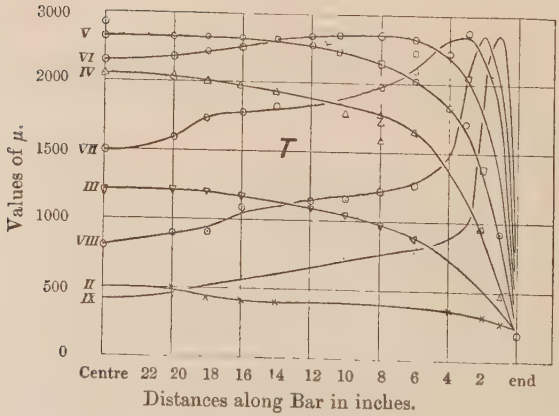

S
Values of H.
40
30
20
10
0
IX
VIII
VII
VI
V
IV
III
II
4
3
2
1
0
Centre 22 20 18 16 14 12 10 8 6 4 2 end
Distances along Bar in inches.

Fig. 8.


T
Values of μ.
3000
2000
1500
1000
500
0
V
VI
IV
VII
III
VIII
II
IX
Centre 22 20 18 16 14 12 10 8 6 4 2 end
Distances along Bar in inches.

practically constant all along (as shown in Q. IX., fig. 4); below such values the demagnetizing force is considerably *larger* than the amount that is usually taken for the length-ratio of 250 to 1 that was used here. Possibly some of this difference is due to imperfect annealing; but great care was taken, and curves P and R agree very well in those parts where from Q. IX. we see the induction is nearly constant along the bar. Furthermore, we should expect that with the great variations shown in set Q the leakage-paths would demand very varying proportions of the impressed magnetomotive force, instead of a constant portion, as is the usual assumption. In fact, we cannot possibly tell what value of B to take in the equation $H=\lambda B$ for demagnetizing force, since in the cylindrical bar (unlike the ellipsoidal) the leakage-flux is not independent of the induction. The ratio of H_m to H_a attains a minimum value near the point of greatest demagnetizing force, *i. e.* about 13,000; but stress cannot be laid on this owing to the reading-off from one curve to the other. It would seem, however, that the assumption of a constant demagnetizing factor is not quite satisfactory; consequently the magnetometric method with cylindrical rods, although extremely useful for comparative work, must be used with much caution in determinations of an absolute character.

On looking at curves Q we note that the induction drops most quickly somewhere in the neighbourhood of *maximum* permeability, that is, the induction leaks out more when the bar is the better conductor. This at first seems peculiar; but on referring to set T, one sees that in the cases where the permeability is increasing as we go along the bar, the induction keeps in the bar, and the quicker the permeability diminishes the more the induction leaks out. In fact, the leakage at any place depends on the average permeability of the part nearer the end of the bar, and not on the value at that particular place.

An interesting relationship is brought out in Table II. It will be noticed that the ratio of B_m and B_h ranges from about 2 per cent. less than unity to about 4 per cent. more. This is not a great variation from unity considering the number of operations to be gone through before arriving at the figures. It would thus appear that the induction in the iron and that

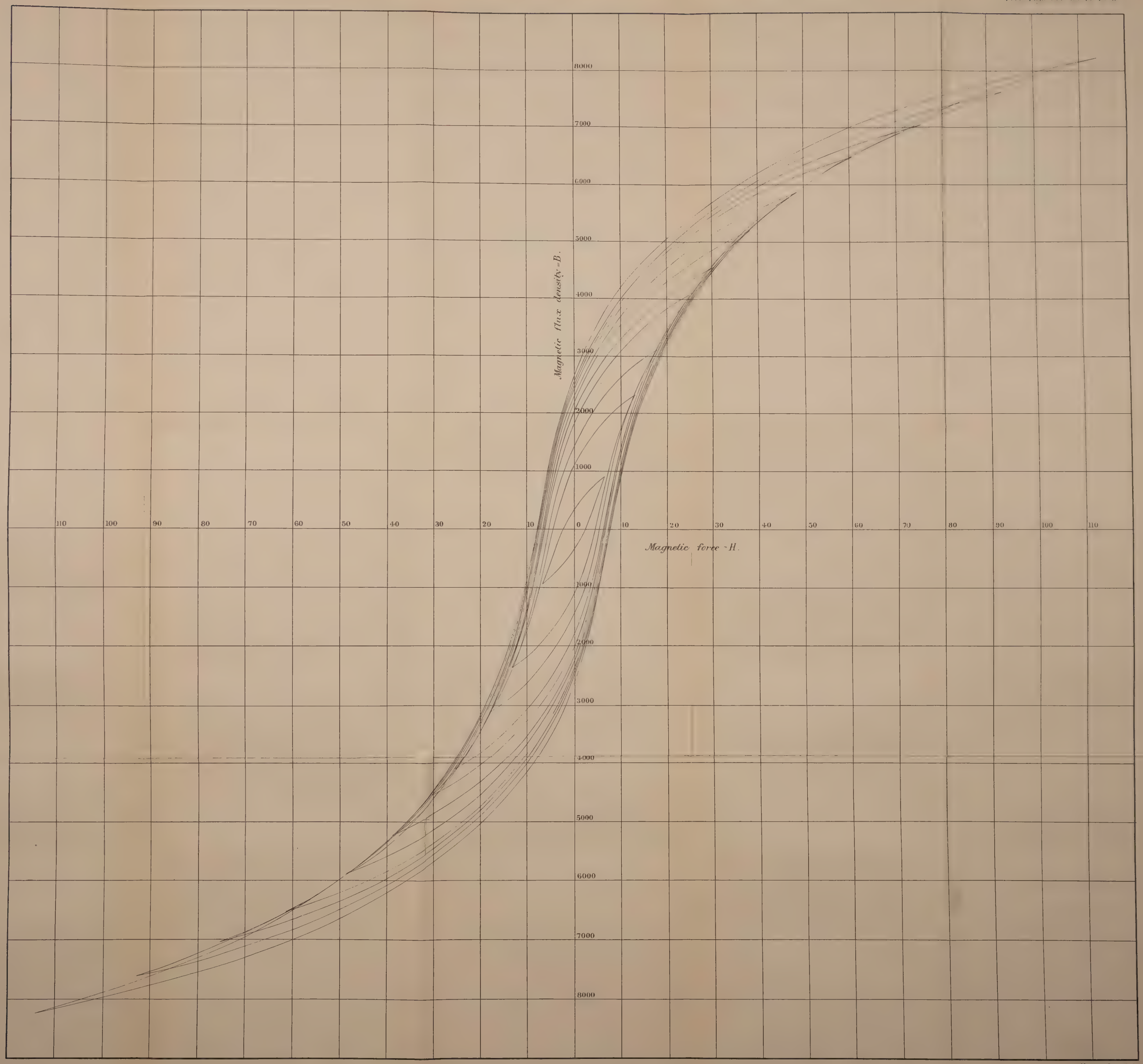
in the air so arrange themselves as to give a mean induction in the bar equal to that which the mean H would have produced had it been applied to the bar and no leakages taken place.

A very striking point is the great alteration in equivalent length of the bar; on the ellipsoidal assumption it is $\frac{2}{3}$ the length, and Kohlrausch's number for a cylinder is $\frac{5}{6}$. We see that, far from this being a constant number, it varies from ·9 to ·74; so that any determination depending on constancy of this quantity will be somewhat vitiated. A direct experiment was made to test the amount of displacement of the poles as follows:—A piece of iron wire was inserted in the magnetizing-solenoid used above, and a brass wire was attached so that it could be slid up or down, always inside the solenoid. It was first adjusted so as to be exactly opposite the needle of a magnetometer. Various magnetizing-currents were then sent round the solenoid, and the wire was moved along inside until the maximum deflexion was produced: the following numbers were obtained:—

Current.	Distance Wire was moved.
1·45	1·5 cm.
1·04	2·2
0·80	4·2
·57	6·5
·45	8·0
·38	12·0
·12	10·5

These results are of course quite rough, but the same general result is obtained as in the first experiments. The direct comparison of a ballistic and a magnetometric curve for the same bar would show that they approximately agree, since the factors causing variation of the equivalent length will affect both determinations in the same way; preliminary experiments have confirmed this. It is hoped to examine bars bent into incomplete rings and other forms.

The kind assistance of Mr. L. G. Walter, B.A., in the earlier experiments is gratefully acknowledged.



SERIES OF HYSTERESIS LOOPS FOR COBALT RING

Fig. 1.

Mantern Bros. lith.

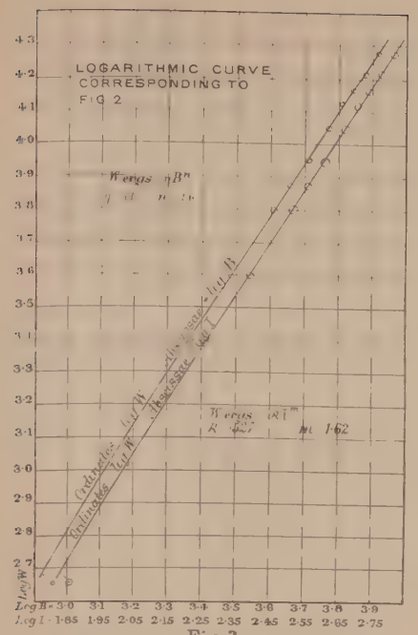


Fig. 3.

Magnetisation Curve of Cobalt

Fig. 4.

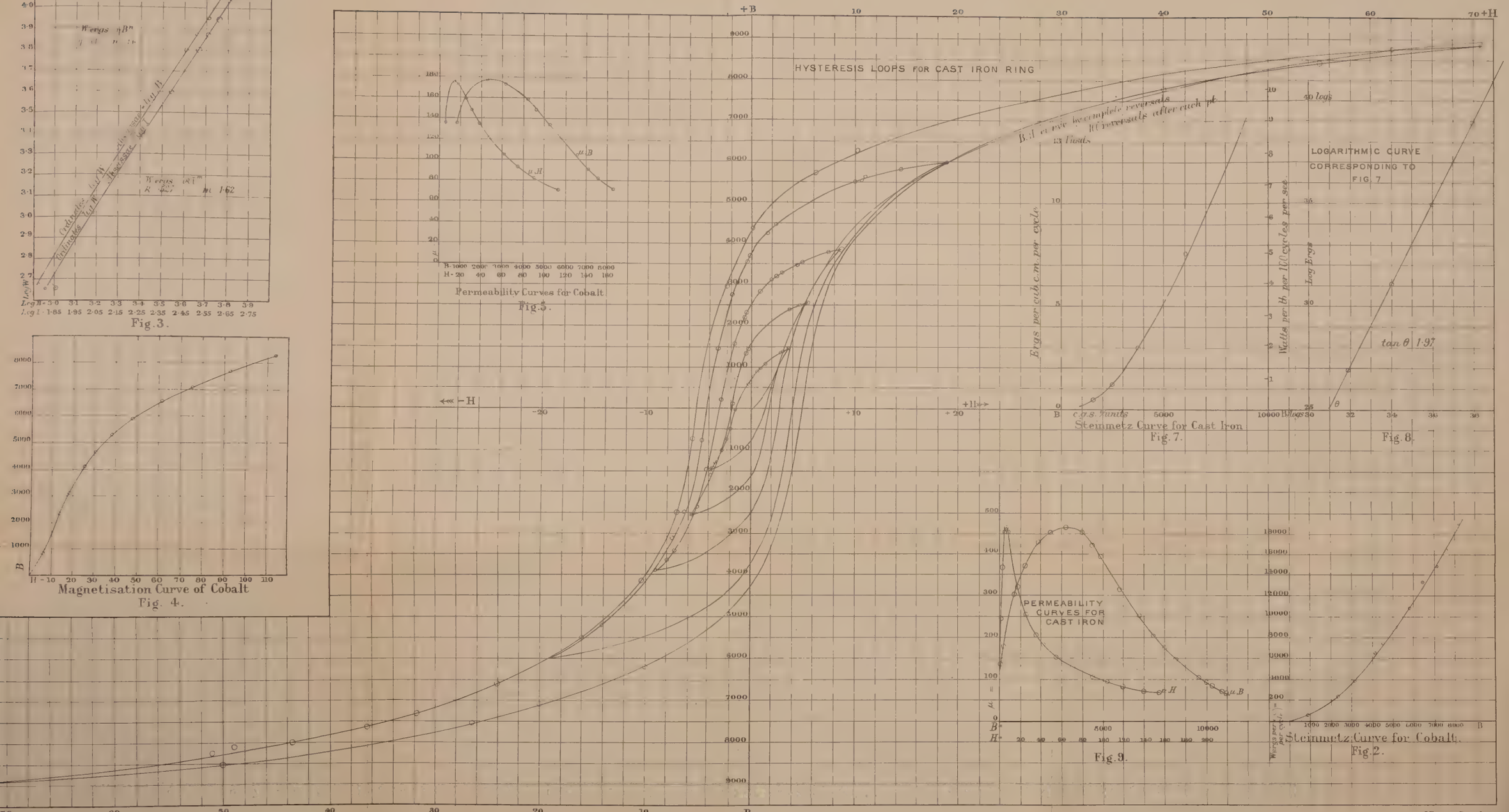


Fig. 6.

Permeability Curves for Cobalt

Fig. 5.

Steinmetz Curve for Cast Iron

Fig. 7.

Fig. 8.

Fig. 9.

Steinmetz Curve for Cobalt.

Fig. 2.

Mintern Bros lith.

LII. *On the Magnetic Hysteresis of Cobalt.* *By* J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London*, A. W. Ashton, *B.Sc.*, 1851 *Exhibition Scholar, and* H. J. Tomlinson, *Salomons Scholar, University College, London**.

[Plates II. & III.]

Although determinations have been made of the magnetic constants of cobalt by other observers, we have not been able to find any very complete set of observations on the magnetic hysteresis values of cobalt of known chemical composition corresponding to various cyclical magnetic forces of known range or maximum value. Having in our possession a cobalt ring of supposed fairly pure commercial cobalt, a series of magnetic experiments were undertaken with it, the results of which are embodied in the following tables and diagrams (see Plates II. & III.).

A rectangular sectioned circular ring of the metal was cast for us by Messrs. H. Wiggin & Co. of Birmingham, and turned up true in the lathe. The dimensions of the ring were then carefully measured and the mean values were found to be as follow :—

Mean outside diameter of ring	13·84 cms.
Mean inside diameter of ring	11·15 „
Diameter of mean perimeter of ring . .	12·50 „
Mean perimeter	39·26 „
Mean depth of cross-section of ring . . .	3·91 „
Mean cross-sectional area	5·23 sq. cm.
The mass of the ring was	1665 grams.
The volume of the ring	205·6 c. cm.
The mean density	8·1

The ring was then insulated with silk tape and wound over with four secondary coils of No. 36 s.w.g. silk-covered copper wire of 100, 50, 25, and 10 turns respectively, put on at quadrantal positions.

* Read June 23, 1899.

Over these secondary coils was wound a series of six primary coils of No. 18 s.w.g. cotton-covered copper wire laid on uniformly round the ring and having 195, 187, 180, 167, 161, and 155 turns respectively. The wire was well insulated with tape and shellac varnish. The ring so prepared was mounted on a board. From some of the turnings a chemical analysis of the metal was made for us by Mr. R. A. Hadfield, M.Inst.C.E., to whom we are greatly indebted.

The analysis showed that the metal had the following composition :—

Cobalt	95·95 per cent.
Nickel	·80 ,,
Iron	·91 ,,
Manganese . . .	·25 ,,
Silicon	·42 ,,
Carbon	1·36 ,,
Sulphur	trace.
Phosphorus . . .	trace.

The metal cannot therefore be considered as even approximately chemically pure. It contains nearly 1 per cent. of iron and 1 per cent. of nickel.

The ring thus prepared was submitted to a complete set of magnetic tests by aid of the ballistic galvanometer in the usual manner for the determination of the magnetic flux-density or induction corresponding to certain cyclic values of magnetizing force. It is unnecessary to give the details, excepting to state that the electrical measurements required were all ultimately referred to a standard resistance and Clark cell by means of a potentiometer. The ballistic galvanometer used was a Crompton-d'Arsonval with a movable coil having a periodic time of fourteen seconds.

The standardization of the galvanometer was effected in the usual manner with a standard secondary coil placed in the interior of a long primary solenoid of known dimensions. From the observations the magnetizing force (H) employed and the corresponding flux-density or induction (B) were

calculated and the results set out in the accompanying tables.

The first set of experiments consisted in taking a careful set of complete magnetic (B.H.) cycles (see fig. 1, Plate II.), and from these the hysteresis losses in ergs per cub. centim. per cycle corresponding to known maximum flux-densities were calculated by measuring the delineated areas. A Steinmetz curve was then constructed showing the relation of the hysteresis loss in each cycle to the maximum induction for that cycle (see fig. 2, Plate III.). This was converted by taking logarithms (ordinary) of both variables and plotting a logarithm curve (see fig. 3). This last curve proved to be nearly a straight line.

If W = magnetic hysteresis energy loss in ergs per cub. centim. per cycle, and
B = maximum flux-density (induction) during the cycle,
I = maximum magnetization during the cycle;

then the logarithmic curves show that over a wide range

$$W = 0{\cdot}01\ B^{1{\cdot}6}, \text{ or}$$
$$W = 0{\cdot}527\ I^{1{\cdot}62}.$$

These exponential expressions $W = \eta B^n$ and $W = \eta' I^{n'}$ for W in terms of B and I show that the relation found by Mr. C. P. Steinmetz* for iron and steels of various composition, and by Dr. A. E. Kennelly† for nickel, approximately holds good for this sample of cobalt, viz. that the hysteresis loss varies as the 1·6th power of the maximum flux-density during the cycle.

It is not a little curious that for materials differing so much as the above cast cobalt and soft annealed transformer iron the hysteretic exponent should in both cases be so near to 1·6. At low inductions the hysteretic exponent increases in value, however, in all cases. In the case of iron it is well known that magnetic hysteresis becomes zero before the maximum induction becomes zero. There is, in fact, a non-hysteretic range of cyclical magnetization. Hence it follows that for some

* 'Electrician,' vol. xxviii. pp. 384, 408, 425; also vol. xxxii. p. 672.
† 'Electrician,' vol. xxviii. p. 666 (1892).

small value of the maximum flux-density there must be a range of flux-density during which hysteresis increases from zero to a finite value; in other words, its rate of change is then very large. Hence $\log_{10} W$ will have a very large negative value corresponding to a certain small value of $\log_{10} B$; and accordingly it follows that the hysteretic constant n, which is represented by the tangent of the inclination of the line (log W, log B) cannot have the value 1·6 throughout all values of $B_{max.}$, but must increase as $B_{max.}$ decreases. That this is also the case for this cobalt sample is well shown by the lowest observation-points on the logarithm-curve (fig. 3). We conclude from the fact that the hysteretic exponent attains a higher value than 1·6 for very low inductions, that there is in cobalt also a non-hysteretic range of cyclical magnetization. It is usual to express W in terms of B; but inasmuch as W is really an expression for the behaviour of the metal *per se* corresponding to a given state of magnetization, it would be better to express W in terms of the magnetization I. Of the three magnetic vectors H, B, and I, I has reference to the properties of the material itself, B to those of the material and the space it occupies as well, and H may be looked upon as denoting the space qualities only.

Hence magnetic material qualities like W should be expressed in terms of I rather than B.

We have accordingly given the logarithmic curve connecting $\log_{10} W$ and $\log_{10} I$, I being the maximum magnetization during the cycle.

The fact that the hysteresis loss W during a cycle can be approximately expressed in terms of the maximum magnetization I during the cycle, seems to show that the work done in making a complete magnetic cycle, or carrying the magnetic molecules once completely round, is nearly a simple exponential function of the percentage of them which are collineated at the beginning and end of the cycle in the direction of maximum magnetization. The magnetization being the magnetic moment per unit volume, it follows that the work done in making a complete magnetic reversal of all the molecular magnets is a nearly simple exponential function

of the total resolved molecular magnetic moments in the direction of magnetization at the beginning and end of the operation.

It appears, therefore, that the work done in carrying the magnetic molecules in unit volume once round a complete cycle is nearly proportional to the 1·6th power of the aggregate magnetic moment of all the molecules estimated in the direction of the magnetic force.

The more the molecular magnets are collineated, that is, the greater the aggregate magnetic moment in a given direction, the greater the work required to effect a complete cyclic operation, the two magnitudes being related by a simple exponential relation. It would be interesting to determine if the mechanical work required to effect the complete magnetic reversal of a crowd of small compass-needles is proportional to the 1·6th power of the aggregate maximum magnetic moment before or after completion of the cycle.

The observations also furnished a simple magnetization-curve for the metal (fig. 4) and a series of values of the permeability stated in terms of the flux-density B and of the magnetic force H (fig. 5).

A study of these curves shows that this cast cobalt magnetically resembles many ordinary qualities of cast iron.

As a comparison, a series of similar observations were taken on a circular cast-iron ring.

This ring was prepared for us by Messrs. Easton, Anderson, and Goolden from a variety of cast iron used by them for dynamo purposes. The transverse section of the ring was rectangular.

The dimensions of the ring were as follows:—

Mean outside diameter . .	5·000 inch.
Mean inside diameter . . .	4·000 „
Depth of ring	0·500 „
Mean thickness	1·000 „
Mass of ring	1·8086 lbs.
Specific gravity	7·042

The ring was insulated and wound over with two primary coils of No. 20 s.w.g. cotton-covered copper wire, the inner layer having 248 turns and the outer 243 turns.

The primary coil was overlaid with two secondary coils of No. 36 silk-covered copper wire, one having 200 turns and the other 20 turns. The ring was submitted to the process above described for obtaining the cyclic magnetization-curves by determining with the ballistic galvanometer the value of B corresponding to various cyclic values of H. These observations were taken by Mr. H. I. Lewenz in January 1898, in the Pender Electrical Laboratory, University College, London.

The hysteresis loops were then carefully plotted out (fig. 6) by the same observer. From the measurements of area a Steinmetz curve was drawn (fig. 7), the ordinates of which represent ergs per cubic centimetre per cycle and the abscissæ maximum flux-density during the cycle. Also a curve (fig. 8) was drawn, the abscissæ and ordinates of which were the ordinary logarithms of the above quantities.

This latter curve proves to be a straight line over a considerable portion of its length. The inclination of the line, however, shows that the hysteretic exponent for this cast iron has a value of 1·97, considerably above the value obtained for the cast cobalt. It appears, therefore, as if the hysteretic exponent for cast iron at moderate and high inductions resembles that of wrought iron, and that of nickel and cobalt at very low induction densities. The permeability curves for this cast iron in terms of B and H as abscissæ are shown in fig. 9.

The chemical composition of the cast-iron ring we employed was kindly determined for us by Mr. R. A. Hadfield. His analysis showed the presence of 2 per cent. of silicon, 2·85 of carbon, and 0·5 of manganese. It was therefore not in any way abnormal. The following tables give an epitome of the observations on these cast cobalt and cast-iron rings.

Observations on the Magnetic Hysteresis of Cast Cobalt.

Cyclical Magnetization Curves.

Loop I.		Loop II.		Loop III.		Loop IV.	
H.	B.	H.	B.	H.	B.	H.	B.
6·67	911	13·23	2341	17·91	3106	25·76	4110
4·92	805	7·13	1928	9·17	2549	14·99	3496
3·24	676	3·75	1595	4·18	2018	7·28	2809
2·28	586	0	1092	0	1436	0	1807
0·99	453	− 3·21	418	− 3·42	727	− 5·28	420
0	347	− 5·36	− 286	− 5·92	− 205	− 8·91	−1005
−0·99	222	− 7·13	− 895	− 8·97	−1288	−12·46	−2023
−2·28	24	− 9·34	−1536	−12·89	−2312	−18·87	−3281
−3·25	−164	−11·72	−2069	−17·91	−3106	−25·76	−4110
−4·92	−537	−13·23	−2341				
−6·67	−911						

Loop V.		Loop VI.		Loop VII.		Loop VIII.	
H.	B.	H.	B.	H.	B.	H.	B.
30·53	4569	38·56	5216	48·54	5869	61·08	6519
17·64	3886	21·15	4397	26·79	4980	36·38	5705
7·75	3047	10·19	3536	10·91	3751	16·39	4440
0	2022	0	2177	0	2299	0	2418
− 5·23	494	− 6·23	201	− 5·60	674	− 6·11	550
− 9·51	−1182	−10·19	−1350	−10·91	−1376	−11·02	−1337
−14·41	−2490	−16·84	−2912	−17·03	−2846	−19·51	−3220
−20·85	−3535	−26·83	−4257	−28·20	−4383	−36·99	−5123
−30·53	−4569	−38·56	−5216	−48·54	−5869	−61·08	−6519

Loop IX.		Loop X.		Loop XI.	
H.	B.	H.	B.	H.	B.
75·46	7052	93·18	7622	114·03	8237
46·41	6224	52·56	6586	60·36	7020
18·43	4684	20·90	4940	24·33	5337
0	2460	0	2521	0	2626
− 6·74	354	− 7·17	240	− 7·86	− 27
−12·86	−1758	−14·97	−2273	− 16·71	−2571
−26·35	−4104	−33·73	−4825	− 37·84	−5167
−46·72	−5786	−57·64	−6366	− 71·10	−6989
−75·46	−7052	−93·18	−7622	−114·03	−8237

Magnetization Curve, Permeability, and Hysteresis Values of Cast Cobalt.

H.	B.	μ.	Hysteresis Loss in ergs per cc. per cycle.	Hysteresis Loss in watts per lb. per 100 cycles per sec.
6·67	911	137	452	·254
13·23	2341	177	2454	1·379
17·91	3106	173	3956	2·222
25·76	4110	160	6292	3·535
30·53	4569	150	7374	4·143
38·56	5216	135	8953	5·029
48·54	5869	121	10937	6·144
61·08	6519	107	13235	7·436
75·46	7052	93	14642	8·226
93·18	7622	82	16518	9·280
114·03	8237	72	18950	10·646

H is the maximum value of the magnetizing force during each cycle.
B is the maximum value of the flux-density or induction.
μ is the value of the permeability corresponding to the annexed values of B, and the hysteretic losses also corresponding to the values of H and B are given in ergs and watt-seconds.

Observations on the Magnetic Hysteresis of Cast Iron.

Magnetization Curve.

H.	B.	μ.	H.	B.	μ.	H.	B.	μ.
·19	27	139	8·84	4030	456	44·65	8071	181
·41	62	150	10·60	4491	424	56·57	8548	151
1·11	206	176	12·33	4884	396	71·98	9097	126
2·53	768	303	13·95	5276	378	88·99	9600	108
3·41	1251	367	15·61	5504	353	103·35	10066	95
4·45	1898	427	18·21	5829	320	120·60	10375	86
5·67	2589	456	26·37	6814	258	140·37	10725	76
7·16	3350	468	36·54	7580	207	152·73	10985	72

Observations on the Magnetic Hysteresis of Cast Iron (*cont.*).

Cyclical Magnetization Curves.

Loop I.

H.	B.	H.	B.	H.	B.	H.	B.
3·80	1475	·78	972	−1·13	240	−3·42	−1357
3·46	1440	·55	886	−1·62	− 103	−3·52	−1384
3·30	1426	·31	845	−2·09	− 494	−3·60	−1413
2·92	1377	0	750	−2·38	− 739	−3·80	−1475
2·82	1351	−·27	663	−2·60	− 906		
2·67	1335	−·30	627	−2·74	−1013		
2·11	1243	−·47	564	−2·86	−1074		
1·32	1090	−·89	374	−3·31	−1290		

Loop II.

H.	B.	H.	B.	H.	B.	H.	B.
5·53	2545	·59	1762	−1·49	730	−5·09	−2348
4·65	2473	·37	1677	−2·33	− 648	−5·19	−2396
3·81	2392	0	1554	−3·39	−1251	−5·53	−2545
3·51	2355	−·27	1421	−3·63	−1469		
2·80	2247	−·37	1388	−3·83	−1642		
2·32	2160	−·59	1285	−4·70	−2150		
1·47	2000	−·85	1133	−4·95	−2286		

Loop III.

H.	B.	H.	B.	H.	B.	H.	B.
8·75	3865	2·75	3228	− ·61	2194	−7·32	−3436
7·46	3781	·89	2799	−1·62	1583	−7·99	−3626
5·19	3569	·38	2605	−2·73	210	−8·75	−3865
4·67	3484	0	2465	−4·15	−1482		
4·17	3441	−·38	2301	−4·97	−2348		

Loop IV.

H.	B.	H.	B.	H.	B.	H.	B.
19·14	5972	1·65	4239	− ·48	3536	− 7·51	−3128
14·77	5825	·89	4047	− ·94	3316	−14·45	−5188
11·38	5646	·47	3950	−1·80	2722	−15·94	−5475
10·92	5526	0	3804	−4·48	− 735	−19·14	−5972
10·21	5496	−·31	3663	−6·36	−2473		

Cyclical Magnetization Curves (*continued*).

Loop V.							
H.	B.	H.	B.	H.	B.	H.	B.
71·50	8930	6·31	5760	−10·52	−4189	−43·47	−7996
43·86	8262	0	4290	−24·10	−6628	−49·00	−8116
32·07	7715	−2·20	2937	−31·75	−7289	−51·11	−8239
10·55	6311	−7·04	−2473	−36·58	−7640	−71·50	−8930

Loop.	B (max.).	Hysteresis Loss.	
		Ergs per cc. per cycle.	Watts per lb. per 100 cycles per sec.
I.	1475	466	·300
II.	2545	1288	·829
III.	3865	2997	1·934
IV.	5972	7397	4·765
V.	8930	13423	8·658

The result of experiments with the cast cobalt ring is to show that although in general form its magnetization-curve resembles that of cast iron, its hysteretic exponent is similar to that of annealed soft iron. The absolute hysteresis values corresponding to various maximum flux-densities are, however, considerably larger than those of a typical variety of cast iron.

Discussion.

Prof. Everett referred to the fact that the sample of cobalt contained about 1 per cent. of iron, and said that it would be interesting to know how cobalt free from iron would behave.

Mr. Blakesley said that the hysteresis curves obtained from the step by step method could not be applied to dynamos, because the time taken to perform the cycle altered the shape of the curve. He would like to see the curves for cobalt determined in cases where the cycles were quickly executed.

LIII. *Demonstration of Prof. T. W. Richards's Method of Standardizing Thermometers.* *By* R. A. Lehfeldt, *D.Sc.**

[Abstract.]

This method depends upon the ordinary latent-heat principle for maintaining constant temperature, but it includes the consideration of the fusion of systems with more than one component. If there are c components and p phases, then the number of degrees of freedom of the system is $[c+2-p]$. When this is zero, the temperature and the pressure of the system are perfectly definite. Thus the following formula represents the four phases to be equilibrated in the case of Glauber's salt:—

$$\left\{\begin{array}{l} Na_2SO_4\,10H_2O \\ Na_2SO_4 \\ \text{Solution} \\ \text{Vapour.} \end{array}\right.$$

The transition point for this system is 32°·379 on the hydrogen scale, and can be determined to within a few thousandths of a degree.

Prof. Richards † has determined the temperature of equilibrium in several useful cases. The salts are put into a test-tube in an air-bath formed between it and a second test-tube; the whole is then heated in a beaker of water over a small flame, or, preferably, in a thermostat kept about $\frac{1}{2}$ degree above the transition point. If the salt is pure, and care is taken to avoid the effect of supersaturation, this method is highly satisfactory. It gives an extensive range of fixed-points, and is especially useful in thermometry for fixed-points between 0° and 100°. A few of these temperatures may be noted:—

Sodium chromate	19°·63 C.
Sodium carbonate	35·2
Sodium thiosulphate . . .	47·9
Sodium bromide	50·7
Manganese chloride . . .	57·7
Trisodium phosphate . . .	73·3
Barium hydroxide	77·9

* Read April 21, 1899.

† *Zeits. f. phys. Chem.* xxvi. p. 690 (1898), xxviii. p. 313 (1899).

Discussion.

Mr. J. A. Harker asked how long the temperature remained constant. Richards's method would appear to be a useful one for standardizing open scale-thermometers, as it would obviate the necessity for the provision of auxiliary bulbs and the 100°-point. It might also be of service as a fixed-point for meteorological thermometers.

Mr. Blakesley said that sodium chromate was represented by a very useful temperature. Could this substance be regarded as sufficiently stable to give a satisfactory fixed-point?

Dr. Lehfeldt, in reply, said that the time taken by the transformation was certainly several hours. All the fixed-points mentioned were theoretically as definite as that corresponding to sodium sulphate, but they had not been so accurately determined.

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